TENSOR CATEGORIES ATTACHED TO DOUBLE GROUPOIDS

NICOLÁS ANDRUSKIEWITSCH AND SONIA NATALE

ABSTRACT. The construction of a quantum groupoid out of a double groupoid satisfying a filling condition and a perturbation datum is given. Several important classes of examples of tensor categories are shown to fit into this construction. Certain invariants such as a pivotal group-like element and quantum and Frobenius-Perron dimensions of simple objects are computed.

CONTENTS

INTRODUCTION

The main goal of this paper is the construction of a large class of examples of weak Hopf algebras, also called *quantum groupoids*, from a certain, quite general, class of double groupoids.

Quantum groupoids have been introduced not long ago [BSz, BNSz]. They are a natural generalization of the notion of a groupoid in a non-commutative context. Heuristically, a finite quantum groupoid consists of two algebra structures in a finite dimensional vector space, subject to a set of compatibility conditions, and possessing an antipode.

Quantum groupoids are interesting objects due to the fact that they give rise, through its representation theory, to rigid tensor categories. Tensor categories are important in several areas of mathematics and physics. In

Date: February 1, 2008.

1991 *Mathematics Subject Classification.* 16W30; 20L05; 18D05; 18B40.

This work was partially supported by CONICET, Fundación Antorchas, Agencia Córdoba Ciencia, ANPCyT and Secyt (UNC).

particular, semisimple finite quantum groupoids give rise to semisimple rigid tensor categories with some finiteness conditions. The key fact about quantum groupoids is that this property does not only make them into a tool for constructing tensor categories, but *every* fusion category is the representation category of a quantum groupoid, in view of results of Hayashi and Ostrik.

A double groupoid is a groupoid object in the category of groupoids. Double groupoids were originally introduced by Ehresmann [E] in the early sixties, and later studied by a number of mathematicians interested in the search of a non commutative relative higher homotopy groupoid of a topological space. Several results on double groupoids have appeared in the literature since then. Remarkably, their connection with crossed modules and a higher Van Kampen Theory has been established in the work of Brown, Higgins, Spencer, *et al.* See for instance the survey paper [B] and references therein.

A double groupoid can be roughly understood as a set of 'boxes' with two groupoid compositions -the *vertical* and *horizontal* compositions-, together with coherent groupoid compositions of the sides, such that the boxes compositions obey a set of compatibility conditions, one of them being the so called *interchange law*.

It seems natural to consider the following construction: given a (finite) double groupoid $\mathcal{T}$, take the vertical and horizontal groupoid algebra structures on the vector space spanned by the boxes of $\mathcal{T}$. This construction was considered by the authors in [AN]; the necessary and sufficient condition for this to produce a quantum groupoid is the *vacancy* of $\mathcal{T}$. The results in [AN] gave a generalization of a celebrated construction in Hopf algebra theory, studied by several people, including G. I. Kac, Takeuchi and Majid. Essentially, a *vacant* double groupoid corresponds to an *exact factorization* of a groupoid. The resulting quantum groupoid is in this case a kind of abelian 'bicrossed product'.

The main result of this paper is the determination of a quantum groupoid structure in the span of the boxes in a double groupoid satisfying a natural *filling condition*, by introducing a certain perturbation in (any) one of the groupoid algebra structures. This is achieved by considering a family of 'corner' functions defined on a double groupoid. In fact, the perturbation is done for a more general class of functions. See Theorems 2.3, 3.6.

Unlike in the vacant context, this more general construction does *not* fit into any known bicrossed product construction coming from matched pairs. It is apparent that our approach is related to the formalism of Ocneanu cells; however, the eventual precise relation still remains to be developed.

We show that several important classes of examples of tensor categories fit into our construction. For instance, for every separable algebra R, the tensor category of R-bimodules (with tensor product $\otimes_R$) is isomorphic to

the category of representation of a (not canonical) quantum groupoid arising from a double groupoid.

The source and target subalgebras of the resulting quantum groupoid turn out to be isomorphic to the groupoid algebra of the *core groupoid* of $\mathcal{T}$. The core groupoid of a double groupoid has been studied by Brown and Mackenzie: it is known that it determines the whole double groupoid, under appropriate restrictions. See [BMa, Theorem 2.7]. However, as pointed out by Brown and Mackenzie in *loc. cit.*, it is not expected that general double groupoids can be described in terms of 'more familiar' structures.

The intrinsic combinatorics of double groupoids reflect into 'Hopf-theoretic' features of the associated quantum groupoids. An instance of this principle is illustrated in Example 2.8, where we show that for any positive rational number r, there exists a double groupoid for which the square of the antipode of the associated weak Hopf algebra has r as one of its eigenvalues.

The paper is organized as follows. Section 1 contains the basic facts about double groupoids needed for the construction. It introduces the corner functions and studies its main properties. Sections 2 and 3 present the main results of the paper: the construction of the quantum groupoid from a double groupoid, and a deformation of this construction via a certain cohomological data. Several properties of these quantum groupoids are also studied in these sections. Finally we consider in Section 4 various examples of tensor categories which fit into our construction.

Conventions. Along this paper, in the case where f, g are composable arrows in a groupoid, their composition will be indicated by juxtaposition from left to right, that is, we shall use the notation fg instead of gf.

For a groupoid $\mathcal{G} \rightrightarrows \mathcal{P}$, with base $\mathcal{P}$, we shall identify $\mathcal{P}$ with a subset of $\mathcal{G}$ via the identity map $\mathcal{G} \to \mathcal{P}$; when there is no ambiguity we shall speak of 'the groupoid $\mathcal{G}$' instead of $\mathcal{G} \rightrightarrows \mathcal{P}$.

Thanks. A preliminary stage of this research was conducted at the University of Miskatonic (Arkham); both authors are grateful to C. D. Ward and H. West for the kind hospitality.

Results of this paper were reported by the first-named author at the Ferrara Algebra Workshop, June 16th-19th 2004. He thanks Claudia Menini for the generous invitation. They were also reported by the second-named author at the International Conference on Quantum Groups, Haifa, 05/07–12/07 2004; she thanks the organizers for the kind invitation and hospitality. Both authors thank the referee for his/her interesting comments.

1. Double groupoids

1.1. **Definition of double groupoid.**

A (finite) *double groupoid* $\mathcal{T}$ is a groupoid object in the category of (finite) groupoids. It is customary to represent a double groupoid in the form of

four related groupoids

$$\begin{array}{ccc} \mathcal{B} & \rightrightarrows & \mathcal{H} \\ \downdownarrows & & \downdownarrows \\ \mathcal{V} & \rightrightarrows & \mathcal{P} \end{array}$$

subject to a set of axioms. See [E, BS]. Throughout this paper, we shall keep the conventions and notations from [AN, Section 2].

The source and target maps of these groupoids are indicated by $t, b : \mathcal{B} \to \mathcal{H}$; $r, l : \mathcal{B} \to \mathcal{V}$; $r, l : \mathcal{H} \to \mathcal{P}$; $t, b : \mathcal{V} \to \mathcal{P}$ ('top', 'bottom', 'right' and 'left'). An element $A \in \mathcal{B}$ is depicted as a box

$$A = l \underset{b}{\overset{t}{\square}} r$$

where $t(A) = t$, $b(A) = b$, $r(A) = r$, $l(A) = l$, and the four vertices of the square representing A are $tl(A) = lt(A)$, $tr(A) = rt(A)$, $bl(A) = lb(A)$, $br(A) = rb(A)$. A box $A \in \mathcal{B}$ is, in general, *not* determined by its boundary.

According to our conventions, horizontal and vertical composition of boxes will be written from left to right and from top to bottom, respectively. We shall write $A|B$ if $r(A) = l(B)$ (so that A and B are horizontally composable), and $\dfrac{A}{B}$ if $b(A) = t(B)$ (so that A and B are vertically composable). The notation AB (respectively, $\frac{A}{B}$) will indicate the horizontal (respectively, vertical) composition; this notation will always implicitly assume that A and B are composable in the appropriate sense. Sometimes we will denote a box having an identity in the top as $\overline{\square}$; or, if the box has an identity in the left, as $\|\square$, etc.

Compositions verify the following. Let $A = l \underset{b}{\overset{t}{\square}} r$ and $B = s \underset{c}{\overset{u}{\square}} m$ in $\mathcal{B}$.

(1.1) If $\quad A|B, \quad$ then $\quad AB = l \underset{bc}{\overset{tu}{\square}} m,$

(1.2) If $\quad \dfrac{A}{B}, \quad$ then $\quad \frac{A}{B} = ls \underset{c}{\overset{t}{\square}} rm.$

The notation $\begin{array}{c|c} A & B \\ \hline C & D \end{array}$ means that all possible horizontal and vertical products are allowed; this implies that $\dfrac{AB}{CD}$, $\frac{A}{C}\Big|\frac{B}{D}$.

Interchange law. If $\begin{array}{c|c} A & B \\ \hline C & D \end{array}$, then

$$AB \atop CD := \begin{Bmatrix} AB \\ CD \end{Bmatrix} = \left\{\begin{matrix} A \\ C \end{matrix}\right\} \left\{\begin{matrix} B \\ D \end{matrix}\right\}. \tag{1.3}$$

The identity functions $\mathbf{id} : \mathcal{H} \to \mathcal{B}$ (vertical identity), $\mathbf{id} : \mathcal{V} \to \mathcal{B}$ (horizontal identity) satisfy

$$\mathbf{id}(x) = \overset{x}{\underset{x}{\boxed{\,\|\quad\|\,}}}, \quad x \in \mathcal{H}; \qquad \mathbf{id}(g) = g \, \boxed{\overline{\underline{\quad}}} \, g, \quad g \in \mathcal{V}.$$

We shall use the notation $\Theta_P := \mathbf{id}\,\mathrm{id}_{\mathcal{H}}\, P = \mathbf{id}\,\mathrm{id}_{\mathcal{V}}\, P$, $\forall P \in \mathcal{P}$.

Suppose that $A = l \overset{t}{\underset{b}{\boxed{\quad}}} r$. The horizontal and vertical inverses of A will be denoted by $A^h = r \overset{t^{-1}}{\underset{b^{-1}}{\boxed{\quad}}} l$, and $A^v = l^{-1} \overset{b}{\underset{t}{\boxed{\quad}}} r^{-1}$, respectively. The element $(A^h)^v = (A^v)^h$ will be denoted A^{-1}; thus $A^{-1} = r^{-1} \overset{b^{-1}}{\underset{t^{-1}}{\boxed{\quad}}} l^{-1}$.

If $\begin{array}{c|c} A & B \\ \hline C & D \end{array}$, then

$$\begin{Bmatrix} AB \\ CD \end{Bmatrix}^h = \begin{matrix} B^h\,A^h \\ D^h\,C^h \end{matrix}, \quad \begin{Bmatrix} AB \\ CD \end{Bmatrix}^v = \begin{matrix} C^v\,D^v \\ A^v\,B^v \end{matrix} \quad \text{and} \quad \begin{Bmatrix} AB \\ CD \end{Bmatrix}^{-1} = \begin{matrix} D^{-1}\,C^{-1} \\ B^{-1}\,A^{-1} \end{matrix}.$$

1.2. **Some properties.**

Let $\mathcal{T}$ be a double groupoid. In this section we collect some technical results from [AN, 1.4] needed later.

Lemma 1.1. [AN, Lemma 1.9]. *Let $A, B, C \in \mathcal{B}$. The following statements are equivalent:*

(i) $ABC \in \mathcal{H}$;

(ii) there exist $U, V \in \mathcal{B}$ such that $\begin{matrix} U \\ V \end{matrix} = B$, $AU \in \mathcal{H}$, $VC \in \mathcal{H}$;

(iii) there exist $W, Z \in \mathcal{B}$ such that $\begin{matrix} W \\ Z \end{matrix} = B$, $AZ \in \mathcal{H}$, $WC \in \mathcal{H}$.

Moreover, in (ii) and (iii) the elements U, V, W, Z are uniquely determined by A, B, C, and we have

$$\begin{array}{c|c|c} A & U & \mathbf{id}\, t(C) \\ \hline \mathbf{id}\, b(A) & V & C \end{array}, \qquad \textit{respectively} \qquad \begin{array}{c|c|c} \mathbf{id}\, t(A) & W & C \\ \hline A & Z & \mathbf{id}\, b(C) \end{array}.$$

$\square$

Dually, we have:

Lemma 1.2. [AN, Lemma 1.10]. *Let $A, B, C \in \mathcal{B}$. The following statements are equivalent:*

(i) $\begin{matrix} A \\ B \\ C \end{matrix} \in \mathcal{V}$;

(ii) there exist $U, V \in \mathcal{B}$ such that $UV = B$, $\begin{matrix} A \\ U \end{matrix} \in \mathcal{V}$, $\begin{matrix} V \\ C \end{matrix} \in \mathcal{V}$;

(iii) there exist $W, Z \in \mathcal{B}$ such that $WZ = B$, $\begin{matrix} A \\ Z \end{matrix} \in \mathcal{V}$, $\begin{matrix} W \\ C \end{matrix} \in \mathcal{V}$.

The elements U, V, W, Z in (ii) and (iii) are uniquely determined by A, B, C, and we have

$$\begin{array}{l|l} A & \mathbf{id}\, r(A) \\ \hline U & V \\ \hline \mathbf{id}\, l(C) & C \end{array}\,, \quad \textit{respectively} \quad \begin{array}{l|l} \mathbf{id}\, l(A) & A \\ \hline W & Z \\ \hline C & \mathbf{id}\, r(C) \end{array}\,.$$

$\square$

Lemma 1.3. [AN, Lemma 1.11]. *(i) Let $A, X, Y, Z \in \mathcal{B}$ such that*

$$\begin{array}{l|l|l} & X^{-1} & \\ \hline X & Y & Z \\ \hline & Z^{-1} & \end{array} \quad . \tag{1.4}$$

Then the following conditions are equivalent:

$$XYZ = A. \tag{1.5}$$

$$\begin{Bmatrix} X^{-1} \\ Y \\ Z^{-1} \end{Bmatrix} = A^{-1}. \tag{1.6}$$

(ii) The collection $X = A = Z$, $Y = A^h$ satisfies (1.4), (1.5) *and* (1.6).

Moreover, if equation (1.4) *holds, then we have also*

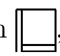
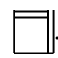
$$\begin{array}{l|l|l} X^v & X^{-1} & A^v \\ \hline X & Y & Z \\ \hline A^v & Z^{-1} & Z^v \end{array} \quad . \tag{1.7}$$

$\square$

1.3. **Core groupoids.**

Let $\mathcal{T}$ be a double groupoid. Brown and Mackenzie have introduced a *core* groupoid, and a related core diagram, relevant to the structure of $\mathcal{T}$. Its relation to the structure of a double Lie groupoid can be found in [Ma2, BMa]. In fact, there are four different core groupoids, all isomorphic via the bijections given by vertical, horizontal and total inversions. In this subsection we recall two of these core groupoids. These will play an important rôle in the description of the source and target subalgebras of the quantum groupoids constructed in Section 2. Let

$$\mathbf{D} := \{D \in \mathcal{B} : \ l(D), b(D) \in \mathcal{P}\},$$
$$\mathbf{E} := \{E \in \mathcal{B} : \ r(E), t(E) \in \mathcal{P}\}.$$

Thus elements of $\mathbf{D}$, resp. of $\mathbf{E}$, are of the form , resp. . Note that $\Theta_P \in \mathbf{D}, \mathbf{E}$, for all $P \in \mathcal{P}$; so $\mathbf{D}, \mathbf{E} \neq \emptyset$.

Proposition 1.4. *There are groupoid structures* $s, e : \mathbf{D} \rightrightarrows \mathcal{P}$, $s, e : \mathbf{E} \rightrightarrows \mathcal{P}$, *with source and target maps* $s(D) = rt(D)$, $e(D) = lt(D)$, $D \in \mathbf{D}$, *resp.* $s(E) = bl(E)$, $e(E) = br(E)$, $E \in \mathbf{E}$, *identity maps* $\mathrm{id} : \mathcal{P} \to \mathbf{D}$, *resp.* $\mathrm{id} : \mathcal{P} \to \mathbf{E}$, $P \mapsto \Theta_P$, *and compositions* $\mathbf{D}\, {}_e\!\times_s \mathbf{D} \to \mathbf{D}$, $\mathbf{E}\, {}_e\!\times_s \mathbf{E} \to \mathbf{E}$, *given by*

$$D \diamond L := \begin{Bmatrix} \mathbf{id}t(L) & D \\ L & \mathbf{id}r(L) \end{Bmatrix}, \qquad M \circ E := \begin{Bmatrix} \mathbf{id}l(E) & E \\ M & \mathbf{id}b(E) \end{Bmatrix}. \tag{1.8}$$

$D, L \in \mathbf{D}$, $M, E \in \mathbf{E}$. *The inverses of* $D \in \mathbf{D}$ *and* $E \in \mathbf{E}$ *are*

$$D^{[-1]} : = (\mathbf{id}t(D)^{-1} D)^v, \tag{1.9}$$

$$E^{(-1)} : = (E\mathbf{id}b(E)^{-1})^v = \begin{Bmatrix} \mathbf{id}l(E)^{-1} \\ E^h \end{Bmatrix}. \tag{1.10}$$

The map $D \mapsto D^{-1}$ *gives an isomorphism of groupoids* $\mathbf{D} \xrightarrow{\simeq} \mathbf{E}$. $\square$

Remark 1.5. $\mathbf{D}$ and $\mathbf{E}$ are *not* subgroupoids of $\mathcal{B}$; in particular, the inverses D^{-1} and $D^{[-1]}$, $D \in \mathbf{D}$, etc., should not be confused. Note that

$$D^\dagger := (D^{[-1]})^{-1} = D^h \mathbf{id}t(D), \qquad D \in \mathbf{D}, \tag{1.11}$$

defines an anti-isomorphism of groupoids, $(\quad)^\dagger : \mathbf{D} \xrightarrow{\simeq} \mathbf{E}$, whose inverse is denoted by the same symbol.

Remark 1.6. If $DE = \mathbf{id}\, u$, for some $D \in \mathbf{D}$, $E \in \mathbf{E}$, $u \in \mathcal{H}$, then $u = t(D)$, $E = D^\dagger$.

Remark 1.7. Recall that the restricted product of $\mathcal{H}^{\mathrm{op}}$ and $\mathcal{V}$ is the groupoid

$$\mathcal{H}^{\mathrm{op}} \boxtimes \mathcal{V} = \{(x, g) \in \mathcal{H} \times \mathcal{V} : l(x) = b(g), \ r(x) = t(g)\},$$

with componentwise multiplication. Then there is a morphism of groupoids

$$\partial : \mathbf{D} \to \mathcal{H}^{\mathrm{op}} \boxtimes \mathcal{V}, \quad \partial(D) = (t(D), r(D)), \quad D \in \mathbf{D}.$$

This morphism can be thought of as the *core diagram* of $\mathcal{T}$ introduced in [BMa, Definition 2.1]; as shown in *loc. cit.*, for all locally trivial double groupoids $\mathcal{T}$ (see Definition 2.9 below), the core diagram of $\mathcal{T}$ determines $\mathcal{T}$.

The kernel of ∂ is the group bundle with $B \in \ker \partial(P)$ whenever its vertical and horizontal sides are the identities of P, $P \in \mathcal{P}$.

There are several canonical maps from certain subsets of $\mathcal{B}$ into $\mathbf{D}$ and $\mathbf{E}$. These arise naturally in the formulas for the source and target maps for the weak Hopf algebras attached to $\mathcal{T}$. They are described in what follows.

The formulas

$$\phi(A) = \begin{Bmatrix} A^{-1} \\ \mathbf{id} r(A) \end{Bmatrix}, \qquad \alpha(A) = \{\mathbf{id} b(A)^{-1} A\},$$

define surjective maps

$$\phi : \{A \in \mathcal{B} :\ t(A) \in \mathcal{P}\} \to \mathbf{D}, \quad \alpha : \{A \in \mathcal{B} :\ l(A) \in \mathcal{P}\} \to \mathbf{D}.$$

Note that if $D \in \mathbf{D}$, then $t(D^{-1}) \in \mathcal{P}$ and $D = \phi(D^{-1}) = \alpha(D)$. On the other hand, the formulas

$$\psi(A) = \begin{Bmatrix} \mathbf{id} l(A) \\ A^{-1} \end{Bmatrix}, \qquad \beta(A) = \{A \mathbf{id} t(A)^{-1}\},$$

define surjective maps

$$\psi : \{A \in \mathcal{B} :\ b(A) \in \mathcal{P}\} \to \mathbf{E}, \quad \beta : \{A \in \mathcal{B} :\ r(A) \in \mathcal{P}\} \to \mathbf{E}.$$

Proposition 1.8. *The groupoid $\mathbf{D}$ acts on the left on the map $rt : \mathcal{B} \to \mathcal{P}$, and on the right on the map $rb : \mathcal{B} \to \mathcal{P}$, by the formulas*

$$D \rightharpoonup A = \begin{Bmatrix} \mathbf{id} t(A) & D \\ A & \mathbf{id} r(A) \end{Bmatrix}, \qquad A \leftharpoonup D = \begin{Bmatrix} A & \mathbf{id} r(A) \\ \mathbf{id} b(A) & \mathbf{id} t(D)^{-1} D \end{Bmatrix}.$$

Dually, $\mathbf{E}$ acts on the right on the map $lt : \mathcal{B} \to \mathcal{P}$, and on the left on the map $lb : \mathcal{B} \to \mathcal{P}$, by the formulas

$$A \leftharpoondown E = \begin{Bmatrix} E \mathbf{id} b(E)^{-1} & \mathbf{id} t(A) \\ \mathbf{id} l(A) & A \end{Bmatrix}, \qquad E \rightharpoondown A = \begin{Bmatrix} \mathbf{id} l(A) & A \\ E & \mathbf{id} b(A) \end{Bmatrix}.$$

□

We remark for future use that there are bijections

$$\{A \in \mathcal{B} :\ l(A) \in \mathcal{P}\} \to \mathcal{H}\, {}_r\!\times_e \mathbf{D}, \quad A \mapsto (b(A), \alpha(A)), \tag{1.12}$$

$$\{B \in \mathcal{B} :\ r(B) \in \mathcal{P}\} \to \mathbf{E}\, {}_e\!\times_l \mathcal{H}, \quad B \mapsto (\beta(B), t(B)), \tag{1.13}$$

the inverses given by horizontal composition.

We next consider the action of the vertical composition groupoid $\mathcal{B} \rightrightarrows \mathcal{H}$ on the map $\gamma : \mathbf{E} \to \mathcal{H}$, $\gamma(E) = b(E)^{-1}$, given by

$$A \curvearrowright E := \psi(E \rightharpoondown A) = \left\{ \begin{matrix} \mathbf{id}\, l(A) \\ E \\ A^{-1} \end{matrix} \right\}, \tag{1.14}$$

$A \in \mathcal{B}$, $E \in \mathbf{E}$, $b(A) = b(E)^{-1}$; the second equality by (1.10).

We consider the following equivalence relation on the groupoid $\mathbf{E}$: we say that E and M in $\mathbf{E}$ are *vertically connected*, denoted $E \sim_V M$ if, and only if, there exists $g \in \mathcal{V}$ such that $t(g) = e(E)$, $b(g) = e(M)$; *i. e.* g connects the ends of E and M.

Remark 1.9. Note that the following conditions are equivalent:

(i) Any two elements of $\mathbf{E}$ are vertically connected;
(ii) $\mathcal{V} \rightrightarrows \mathcal{P}$ is connected.

Indeed, the implication (ii) $\Rightarrow$ (i) is evident, while (i) $\Rightarrow$ (ii) follows from vertical connectedness of the boxes $\Theta_P \in \mathbf{E}$, $P \in \mathcal{P}$.

Lemma 1.10. *Let E and $M \in \mathbf{E}$. Then $E \sim_V M$ if and only if there exists $A \in \mathcal{B}$ such that $b(E)b(A) \in \mathcal{P}$ and $M = A \curvearrowright E$.*

Proof. If $M = A \curvearrowright E$, then $l(A) \in \mathcal{V}$ has source $e(M)$ and target $e(E)$; hence $E \sim_V M$. Conversely, assume that there exists $g \in \mathcal{V}$ such that $t(g) = e(E)$, $b(g) = e(M)$. Then

$$A = \left\{ \begin{matrix} M^{-1} \\ \mathbf{id} g^{-1} \\ E^h \end{matrix} \right\}$$

satisfies $M = \psi(E \rightharpoondown A) == A \curvearrowright E$. $\square$

Finally, the maps ϕ and ψ play also a rôle in the next result needed later.

Lemma 1.11. *Let $A \in \mathcal{B}$. Then we have*

(i) There exist $X, Y \in \mathcal{B}$ such that $\begin{array}{c|c} & A \\ \hline X & Y \end{array}$, $XY \in \mathcal{H}$, $\begin{matrix} A \\ Y \end{matrix} \in \mathcal{V}$, if and only if $t(A) \in \mathcal{P}$. In this case, we have $Y = \phi(A)^h$ and $X = \mathbf{id} x\, \phi(A)$, for a unique $x \in \mathcal{H}$ such that $r(x) = rb(A)$.

(ii) There exist $X, Y \in \mathcal{B}$ such that $\begin{array}{c|c} X & Y \\ \hline A & \end{array}$, $XY \in \mathcal{H}$, $\begin{matrix} X \\ A \end{matrix} \in \mathcal{V}$, if and only if $b(A) \in \mathcal{P}$. In this case, we have $X = \psi(A)^h$ and $Y = \psi(A)\, \mathbf{id} y$, for a unique $y \in \mathcal{H}$ such that $l(y) = bl(A)$.

Proof. (i) It is clear that if there is such a pair X, Y then $t(A)$ is an identity. Suppose that this is the case, and let X, Y as in (i). Since $XY = \mathbf{id}t(XY)$, then $r(Y) = \operatorname{id} tr(Y)$, and thus $\begin{matrix} A \\ Y \end{matrix} = \mathbf{id}r(A)$; on the other hand, since $\begin{matrix} A \\ Y \end{matrix}$ is an identity, then $b(Y) = \operatorname{id} lb(Y)$, and therefore $X = \mathbf{id}b(X)Y^h = \mathbf{id}b(X)\left\{\begin{matrix} A^{-1} \\ \mathbf{id}r(A) \end{matrix}\right\}$. Note in addition that $rb(X) = br(A)$. Part (i) will be established if we prove that for any $x \in \mathcal{H}$ with $r(x) = br(A)$ there is an X as in (i) with $b(X) = x$. This is done by letting $X = \mathbf{id}x\left\{\begin{matrix} A^{-1} \\ \mathbf{id}r(A) \end{matrix}\right\}$. We omit the proof of part (ii), which is similar. $\square$

1.4. **Corner functions.**

We begin by introducing four 'corner' maps on the set of boxes. These will prove useful later in order to define the coproduct of a related quantum groupoid. Let

$$\ulcorner : \mathcal{V}_t \times_l \mathcal{H} \to \mathbb{N} \cup \{0\}, \quad \llcorner : \mathcal{V}_b \times_l \mathcal{H} \to \mathbb{N} \cup \{0\},$$
$$\urcorner : \mathcal{V}_t \times_r \mathcal{H} \to \mathbb{N} \cup \{0\}, \quad \lrcorner : \mathcal{V}_b \times_r \mathcal{H} \to \mathbb{N} \cup \{0\},$$

be given by the formulas:

$$\ulcorner(g, x) = \#\left\{U \in \mathcal{B} : U = \begin{array}{cc} & x \\ g & \square \end{array} \right\}; \quad \urcorner(g, x) = \#\left\{U \in \mathcal{B} : U = \begin{array}{cc} x & \\ \square & g \end{array} \right\};$$

$$\llcorner(g, x) = \#\left\{U \in \mathcal{B} : U = \begin{array}{cc} g & \square \\ & x \end{array} \right\}; \quad \lrcorner(g, x) = \#\left\{U \in \mathcal{B} : U = \begin{array}{cc} \square & g \\ x & \end{array} \right\}.$$

Thus $\ulcorner(g, x)$ equals the number of boxes $B \in \mathcal{B}$ with $l(B) = g$, $t(B) = x$, and so on.

These determine four maps $\ulcorner, \llcorner, \urcorner, \lrcorner : \mathcal{B} \to \mathbb{N}$ by the following rules:

$$\ulcorner(X) = \ulcorner(l(X), t(X)); \quad \urcorner(X) = \urcorner(r(X), t(X));$$
$$\llcorner(X) = \llcorner(l(X), b(X)); \quad \lrcorner(X) = \lrcorner(r(X), b(X)).$$

Lemma 1.12. *(i) Let $x \in \mathcal{H}$, $g \in \mathcal{V}$. Then $\ulcorner(g, x) = \llcorner(g^{-1}, x) = \urcorner(g, x^{-1}) = \lrcorner(g^{-1}, x^{-1})$.*

(ii) Let $X \in \mathcal{B}$. Then $\ulcorner(X) = \llcorner(X^v) = \urcorner(X^h) = \lrcorner(X^{-1})$.

Proof. Part (i) is an easy consequence of the definitions, using the bijections given by vertical, horizontal and total inversions in $\mathcal{T}$.

Part (ii) follows from (i). $\square$

Definition 1.13. Let $A, B, X, Y \in \mathcal{B}$ such that $XY = \genfrac{}{}{0pt}{}{A}{B}$. By a *double factorization* of this common product we shall mean a quadruple (U, V, R, S) of elements in $\mathcal{B}$ satisfing

$$\begin{array}{c|c} U & V \\ \hline R & S \end{array}, \quad UV = A, \quad RS = B, \quad \genfrac{}{}{0pt}{}{U}{R} = X, \quad \genfrac{}{}{0pt}{}{V}{S} = Y. \tag{1.15}$$

The set of all double factorizations will be denoted $[X, Y, A, B]$.

Proposition 1.14. *Let $A, B, X, Y \in \mathcal{B}$ such that $XY = \genfrac{}{}{0pt}{}{A}{B}$. Then we have*

$$\begin{aligned}\#[X, Y, A, B] &= \ulcorner(l(A), t(X)) = \llcorner(l(B), b(X)) \\ &= \urcorner(r(A), t(Y)) = \lrcorner(r(B), b(Y)).\end{aligned}$$

Proof. The map $[X, Y, A, B] \to \left\{U \in \mathcal{B} : U = {}_{l(A)}\overset{t(X)}{\square}\right\}$, given by $(U, V, R, S) \mapsto U$ is a well defined bijection, whose inverse is given by $U \mapsto \left(U, U^h A, \genfrac{}{}{0pt}{}{U^v}{X}, \genfrac{}{}{0pt}{}{U^{-1}A^v}{Y}\right)$. This shows the first equality. The others are similarly established. $\square$

As a consequence we get the following symmetry properties of the corner maps:

Corollary 1.15. *Let $L, M, N \in \mathcal{B}$. Suppose that $\begin{array}{c|c} L & M \\ \hline N & \end{array}$. Then we have*

$$\begin{aligned} &(i) \quad \ulcorner(L) = \urcorner(M), && (ii) \quad \llcorner(L) = \lrcorner(M), \\ &(iii) \quad \ulcorner(L) = \llcorner(N), && (iv) \quad \urcorner(L) = \lrcorner(N). \end{aligned}$$

In particular, $\ulcorner(L) = \urcorner(L)$, $\llcorner(L) = \lrcorner(L)$, for every horizontal identity $L \in \mathcal{V}$, and $\ulcorner(N) = \llcorner(N)$, $\urcorner(N) = \lrcorner(N)$, for every vertical identity $N \in \mathcal{H}$.

Proof. The quadruple $X = L$, $Y = M$, $A = LM$, $B = \operatorname{id} b(LM)$ satisfies the assumptions of Proposition 1.14, and then the proposition implies that

$$\ulcorner(l(L), t(L)) = \ulcorner(l(A), t(X)) = \urcorner(r(A), t(Y)) = \urcorner(r(M), t(M));$$

this proves part (i). Part (ii) follows similarly, considering instead the set $[L, M, \mathbf{id}\, t(LM), LM]$. As to parts (iii) and (iv), the same arguments apply with $A = L$, $B = N$, $X = \genfrac{}{}{0pt}{}{L}{N}$, $Y = \operatorname{id} r \begin{pmatrix} L \\ N \end{pmatrix}$. $\square$

Lemma 1.16. *Let $P \in \mathcal{P}$. Then we have*

$$\lrcorner(\operatorname{id}_{\mathcal{V}} P, \operatorname{id}_{\mathcal{H}} P) = \urcorner(\operatorname{id}_{\mathcal{V}} P, \operatorname{id}_{\mathcal{H}} P) = \llcorner(\operatorname{id}_{\mathcal{V}} P, \operatorname{id}_{\mathcal{H}} P) = \ulcorner(\operatorname{id}_{\mathcal{V}} P, \operatorname{id}_{\mathcal{H}} P).$$

The common value in Lemma 1.16 will be denoted $\theta(P)$. This agrees with the value of any of the corner functions on the box Θ_P.

Proof. The proof follows from Lemma 1.12. □

A surprising consequence of Proposition 1.14 is that the corner functions on a box actually depend only on one vertex, the vertex 'opposite' to the corner, of that box.

Proposition 1.17. *Let $L \in \mathcal{B}$. Then the following hold.*

$$(i)\ \ulcorner(L) = \theta(br(L)), \qquad (ii)\ \llcorner(L) = \theta(rt(L)),$$
$$(iii)\ \urcorner(L) = \theta(bl(L)), \qquad (iv)\ \lrcorner(L) = \theta(tl(L)).$$

Proof. We show part (i); then parts (ii)–(iv) follow from (i) and Lemma 1.12. To do this we argue as in the proof of Corollary 1.15. Let $X = L$, $Y = \mathbf{id}r(L)$, $A = L$, $B = \mathbf{id}b(L)$. By Proposition 1.14,

$$\ulcorner(L) = \ulcorner(l(A), t(X)) = \lrcorner(r(B), b(Y)) = \lrcorner(id_{\mathcal{V}} br(L), \mathrm{id}_{\mathcal{H}}\, br(L)) = \theta(br(L)),$$

as claimed. □

Lemma 1.18. *Let $P, Q \in \mathcal{P}$. Suppose that P and Q are connected by an arrow of the core groupoid $\mathbf{D}$. Then $\theta(P) = \theta(Q)$.*

Note that P and Q are connected by $\mathbf{D}$ if and only if they are connected by $\mathbf{E}$.

Proof. Let $[P]$ denote the connected component of $\mathcal{P}$ with respect to $\mathbf{D}$ containing P. We have

$$\begin{aligned}\theta(P) &= \sum_{R \in \mathcal{P}} \#\{B \in \mathcal{B} :\ l(B) = \mathrm{id}_P,\, b(B) = \mathrm{id}_P,\, rt(B) = R\} \\ &= \sum_{R \in \mathcal{P}} \#\mathbf{D}(R, P) = \mathbf{D}(P)\ \#[P].\end{aligned}$$

Thus, $\theta(P) = \theta(Q)$, whenever they are in the same connected component with respect to $\mathbf{D}$. □

The following proposition states the main translation invariance properties of the above defined maps.

Proposition 1.19. *Let $X, Y, Z \in \mathcal{B}$ such that $\begin{array}{c|c} X & Y \\ \hline Z & \end{array}$. Then we have*

$$(i) \quad \urcorner(XY) = \urcorner(X), \qquad (ii) \quad \urcorner\begin{pmatrix} X \\ Z \end{pmatrix} = \urcorner(Z),$$
$$(iii) \quad \llcorner(XY) = \llcorner(Y), \qquad (iv) \quad \llcorner\begin{pmatrix} X \\ Z \end{pmatrix} = \llcorner(X).$$

Similar properties hold for the functions $\lrcorner$ and $\ulcorner$.

Proof. We prove (i):

$$\urcorner(XY) = \theta(bl(XY)) = \theta(bl(X)) = \urcorner(X),$$

by Proposition 1.17. The proof of parts (ii), (iii) and (iv) is similar. $\square$

1.5. Counting formulas in double groupoids.

We obtain in this subsection some counting formulas, involving the corner functions, that will be of use in the next section.

Lemma 1.20. *Let $A \in \mathcal{B}$. There is a bijection between*

$$\big\{(X,Y,Z) \in \mathcal{B}^3 : (X,Y,Z) \textit{ satisfies } (1.4), (1.5) \textit{ in } 1.3\big\}$$

and $\Big\{X \in \mathcal{B} : X = {}_{l(A)}\underset{b(A)}{\square}\Big\} \times \Big\{Z \in \mathcal{B} : Z = \overset{t(A)}{\square}{}_{r(A)}\Big\}$. *In particular,*

$$\#\big\{(X,Y,Z) \in \mathcal{B}^3 : (X,Y,Z) \textit{ satisfies } (1.4), (1.5) \textit{ in } 1.3\big\} = \llcorner(A)\urcorner(A).$$

Proof. For any such triple we have $X = {}_{l(A)}\underset{b(A)}{\square}$ and $Z = \overset{t(A)}{\square}{}_{r(A)}$. Moreover, Y is determined by $Y = X^h A Z^h$. Conversely, for every pair (X,Z) as above, the triple $(X, X^h A Z^h, Z)$ satisfies (1.4), (1.5). This proves the lemma. $\square$

Lemma 1.21. *Let $A, X, Y \in \mathcal{B}$. Assume that $tr(A) = lb(Y)$ and $bl(A) = rt(X)$. Then the following hold:*

$$\#\Big\{(U,V) \in \mathcal{B}^2 : UV = A,\quad \begin{smallmatrix} U \\ V^{-1} \end{smallmatrix} = Y\Big\} = \begin{cases} \urcorner(r(A), b(Y)^{-1}) = \ulcorner(l(A), t(Y)) & \textit{if } b(A) \in \mathcal{P}, \\ 0 & \textit{otherwise.} \end{cases} \tag{1.16}$$

$$\#\Big\{(U,V) \in \mathcal{B}^2 : UV = A,\quad \begin{smallmatrix} U^{-1} \\ V \end{smallmatrix} = X\Big\} = \begin{cases} \llcorner(l(A), t(X)^{-1}) = \lrcorner(r(A), b(X)) & \textit{if } t(A) \in \mathcal{P}, \\ 0 & \textit{otherwise.} \end{cases} \tag{1.17}$$

Proof. The set of pairs (U,V) as in (1.16) coincides with the set of pairs (U,V) satisfing

$$\begin{array}{c|c} U & V \\ \hline V^{-1} & V^v \end{array}, \quad UV = A, \quad \begin{matrix} U \\ V^{-1} \end{matrix} = Y,$$

which is in bijective correspondence with the set of quadruples (U, V, R, S) such that

$$\begin{array}{c|c} U & V \\ \hline R & S \end{array}, \quad UV = A, \quad \begin{matrix} U \\ R \end{matrix} = Y, \quad RS = \mathbf{id}r(A)^{-1}, \quad \begin{matrix} V \\ S \end{matrix} = \mathbf{id}b(Y)^{-1},$$

by sending (U, V) to (U, V, V^{-1}, V^{v}), and (U, V, R, S) to (U, V). Hence Equation (1.16) follows from Proposition 1.14. Equation (1.17) is similarly shown. $\square$

1.6. **Vacant double groupoids.**

A double groupoid $\mathcal{T}$ is called *vacant* if for any $g \in \mathcal{V}$, $x \in \mathcal{H}$ such that $r(x) = t(g)$, there is exactly one $X \in \mathcal{B}$ such that $X = \begin{matrix} x & \\ \square & g \end{matrix}$. Vacant double groupoids have been introduced in [Ma1, Definition 2.11]. We gave in [AN] several characterizations of vacant double groupoids that we had found in the course of our research; the following is an example of a completely symmetric one. Its proof is a direct consequence of Proposition 1.14.

Proposition 1.22. [AN]. *Let $\mathcal{T}$ be a double groupoid. The following are equivalent.*

(1) *$\mathcal{T}$ is vacant.*
(2) *For all $A, B, X, Y \in \mathcal{B}$ such that $XY = \begin{matrix} A \\ B \end{matrix}$, there exist unique U, V, R, $S \in \mathcal{B}$ satisfing Equation 1.15.*

$\square$

In terms of corner functions, vacant double groupoids are characterized by the property that some (hence all) corner function $\mathcal{V}_{t/b} \times_{l/r} \mathcal{H} \to \mathbb{N} \cup \{0\}$ takes constantly the value 1.

If $\mathcal{T}$ is vacant, the core groupoids $\mathbf{D}$ and $\mathbf{E}$ are isomorphic and coincide with the discrete groupoid on the base $\mathcal{P}$: that is, the only arrows in the core groupoids are the identity arrows. In fact, we have the following characterization.

Proposition 1.23. *Let $\mathcal{T}$ be a double groupoid. The following are equivalent.*

(1) *$\mathcal{T}$ is vacant.*
(2) *Some (hence all) corner function takes positive values*[1] *and the core groupoids are discrete on the base $\mathcal{P}$.*

[1]This is the filling condition (2.10) below.

Proof. We have already discussed $1 \implies 2$.

$2 \implies 1$. Since $\mathbf{E} \simeq \mathcal{P}$, $\theta(P) = 1$ for all $P \in \mathcal{P}$. If $(g, x) \in \mathcal{V}_t \times_r \mathcal{H}$ then there exists at least one box $A \in \mathcal{B}$ with $t(A) = x$, $r(A) = g$. Let $P = bl(A)$. Then $\urcorner(g, x) = \urcorner(A) = \theta(P) = 1$, the second equality by Proposition 1.17 (iii). □

2. Construction of quantum groupoids from double groupoids

2.1. Quantum groupoids.

Recall [BNSz, BSz] that a *weak bialgebra* structure on a vector space H over a field $\Bbbk$ consists of an associative algebra structure $(H, m, 1)$, a coassociative coalgebra structure (H, Δ, ϵ), such that the following are satisfied:

$$\Delta(ab) = \Delta(a)\Delta(b), \qquad \forall a, b \in H. \tag{2.1}$$

$$\Delta^{(2)}(1) = (\Delta(1) \otimes 1)(1 \otimes \Delta(1)) = (1 \otimes \Delta(1))(\Delta(1) \otimes 1). \tag{2.2}$$

$$\epsilon(abc) = \epsilon(ab_1)\epsilon(b_2c) = \epsilon(ab_2)\epsilon(b_1c), \qquad \forall a, b, c \in H. \tag{2.3}$$

The maps ϵ_s, ϵ_t given by

$$\epsilon_s(h) = (\mathrm{id} \otimes \epsilon)((1 \otimes h)\Delta(1)),$$
$$\epsilon_t(h) = (\epsilon \otimes \mathrm{id})(\Delta(1)(h \otimes 1)),$$

$h \in H$, are respectively called the source and target maps; their images are respectively called the source and target subalgebras.

A weak bialgebra H is called a *weak Hopf algebra* or a *quantum groupoid* if there exists a linear map $\mathcal{S} : H \to H$ satisfying

$$m(\mathrm{id} \otimes \mathcal{S})\Delta(h) = \epsilon_t(h), \tag{2.4}$$

$$m(\mathcal{S} \otimes \mathrm{id})\Delta(h) = \epsilon_s(h), \tag{2.5}$$

$$m^{(2)}(\mathcal{S} \otimes \mathrm{id} \otimes \mathcal{S})\Delta^{(2)} = \mathcal{S}, \tag{2.6}$$

for all $h \in H$. See [NV] for a survey on quantum groupoids. It is known that a weak Hopf algebra is a true Hopf algebra if and only if $\Delta(1) = 1 \otimes 1$.

2.2. Weak Hopf algebras arising from double groupoids.

Let $\mathcal{T}$ be a *finite* double groupoid, that is, $\mathcal{B}$ is a finite set (and *a fortiori* also $\mathcal{V}$, $\mathcal{H}$ and $\mathcal{P}$ are finite).

Let $\Bbbk$ be a field of characteristic zero and let $\Bbbk\mathcal{T}$ denote the $\Bbbk$-vector space with basis $\mathcal{B}$. We define a multiplication and comultiplication on $\Bbbk\mathcal{T}$ by the formulas

$$A.B = \begin{cases} \dfrac{A}{B}, & \text{if } \dfrac{A}{B}, \\ 0, & \text{otherwise,} \end{cases} \tag{2.7}$$

$$\Delta(A) = \sum_{XY=A} \frac{1}{\urcorner(Y)}\, X \otimes Y = \sum_{XY=A} \frac{1}{\ulcorner(X)}\, X \otimes Y, \tag{2.8}$$

for all $A, B \in \mathcal{B}$. The second identity because of Corollary 1.15.

Therefore $\Bbbk\mathcal{T}$ is an associative algebra with unit $\mathbf{1} := \sum_{x\in\mathcal{H}} \mathbf{id}\, x$. This algebra structure coincides with the groupoid algebra structure on $\Bbbk\mathcal{T}$ corresponding to the vertical composition groupoid $\mathcal{B} \rightrightarrows \mathcal{H}$. See [AN].

The coalgebra structure on $\Bbbk\mathcal{T}$ is a modification of the dual groupoid coalgebra of the horizontal composition groupoid $\mathcal{B} \rightrightarrows \mathcal{V}$, studied in [AN].

Lemma 2.1. *The comultiplication* (2.8) *makes* $\Bbbk\mathcal{T}$ *into a coassociative coalgebra, with counit* $\epsilon : \Bbbk\mathcal{T} \to \Bbbk$ *given by*

$$\epsilon(A) = \begin{cases} \ulcorner(A) = \urcorner(A), & \textit{if } A \in \mathcal{V}, \\ 0, & \textit{otherwise.} \end{cases}$$

Proof. Let $A \in \mathcal{B}$. We have

$$(\Delta \otimes \mathrm{id})\Delta(A) = \sum_{XYZ=A} \frac{1}{\urcorner(Y)\urcorner(Z)}\, X \otimes Y \otimes Z,$$
$$(\mathrm{id} \otimes \Delta)\Delta(A) = \sum_{XYZ=A} \frac{1}{\urcorner(YZ)\urcorner(Z)}\, X \otimes Y \otimes Z.$$

Thus coassociativity of Δ follows from Proposition 1.19. The counit axiom $(\mathrm{id} \otimes \epsilon)\Delta = \mathrm{id}$ is straightforward to check. Also, using the definitions of Δ and ϵ, we have for every $A \in \mathcal{B}$,

$$(\epsilon \otimes \mathrm{id})\Delta(A) = \sum_{XY=A} \frac{1}{\ulcorner(X)}\, \epsilon(X) Y = \frac{\urcorner(l(A))}{\ulcorner(l(A))}\, A = A;$$

the last identity in view of Corollary 1.15. $\square$

Recall from Subsection 1.3 the definition of the core groupoids $\mathbf{D}$ and $\mathbf{E}$. Let $D \in \mathbf{D}$, $E \in \mathbf{E}$. We introduce elements ${}_D\mathbf{1}, \mathbf{1}_E \in \Bbbk\mathcal{T}$ by

$${}_D\mathbf{1} := \sum_{z\in\mathcal{H},\, r(z)=e(D)} \{\mathbf{id}z\, D\}, \quad \mathbf{1}_E := \sum_{x\in\mathcal{H},\, l(x)=e(E)} \{E\, \mathbf{id}x\}. \tag{2.9}$$

Observe that the maps $D \mapsto {}_D\mathbf{1}$, $E \mapsto \mathbf{1}_E$, are both injective.

We relate the groupoid structures of $\mathbf{D}$, $\mathbf{E}$ with the multiplication in $\Bbbk\mathcal{T}$.

Lemma 2.2. *Let $D, L \in \mathbf{D}$, $E, M \in \mathbf{E}$. We have*

$$
{}_D\mathbf{1}.{}_L\mathbf{1} = \begin{cases} {}_{D\diamond L}\mathbf{1}, & \textit{if } e(D) = s(L), \\ 0, & \textit{otherwise}; \end{cases}
$$

$$
\mathbf{1}_E.\mathbf{1}_M = \begin{cases} \mathbf{1}_{M\circ E}, & \textit{if } e(M) = s(E), \\ 0, & \textit{otherwise}. \end{cases}
$$

Proof. We compute

$$
\begin{aligned}
{}_D\mathbf{1}.{}_L\mathbf{1} &= \sum_{\substack{r(z) = e(D), \\ r(w) = e(L)}} \{\mathbf{id}z\, D\}.\{\mathbf{id}w\, L\} = \sum_{\substack{r(z) = e(D), \\ r(w) = e(L), \\ z = wt(L)}} \begin{Bmatrix} \mathbf{id}z\, D \\ \mathbf{id}w\, L \end{Bmatrix} \\
&= \delta_{e(D),s(L)} \sum_{r(w)=e(L)} \begin{Bmatrix} \mathbf{id}w\, \mathbf{id}t(L)\, D \\ \mathbf{id}w\, L \end{Bmatrix} \\
&= \delta_{e(D),s(L)} \sum_{r(w)=e(L)} \mathbf{id}w\, (D \diamond L) = \delta_{e(D),s(L)}\, {}_{D\diamond L}\mathbf{1}.
\end{aligned}
$$

This proves the first claim. The second is similarly shown. □

In what follows we shall consider double groupoids $\mathcal{T}$ which satisfy the following *filling condition*:

$$
\urcorner(g, x) \neq 0, \qquad \forall x \in \mathcal{H},\ g \in \mathcal{V},\ r(x) = t(g). \tag{2.10}
$$

This filling condition on double groupoids has been considered by Mackenzie [Ma2]. It is easy to exhibit examples of double groupoids that do not satisfy the filling condition; *e. g.* the union of the vertical and horizontal subgroupoids of a suitable double groupoid. There are three other equivalent formulations of (2.10) in terms of the other corners, *cf.* Lemma 1.12.

Theorem 2.3. *Suppose that $\mathcal{T}$ is a double groupoid satisfying* (2.10)*. Then $\Bbbk\mathcal{T}$ is a weak Hopf algebra with multiplication and comultiplication given by* (2.7) *and* (2.8)*. The antipode is determined by the formula*

$$
\mathcal{S}(A) = \frac{\ulcorner(A)}{\llcorner(A)}\, A^{-1}, \tag{2.11}
$$

for all $A \in \mathcal{B}$. The source and target maps are given, respectively, by

$$\epsilon_s(A) = \begin{cases} {}_{\phi(A)}\mathbf{1}, & \text{if} \quad t(A) \in \mathcal{P}, \\ 0, & \text{otherwise}; \end{cases} \tag{2.12}$$

$$\epsilon_t(A) = \begin{cases} \dfrac{\ulcorner(A)}{\llcorner(A)}\, \mathbf{1}_{\psi(A)}, & \text{if} \quad b(A) \in \mathcal{P}, \\ 0, & \text{otherwise}. \end{cases} \tag{2.13}$$

The source and target subalgebras are isomorphic to the groupoid algebras $\Bbbk\mathbf{D}$ and $(\Bbbk\mathbf{E})^{\mathrm{op}}$, respectively.

Here, the maps ϕ, ψ are those defined in Subsection 1.3.

Remarks 2.4. (i). The weak Hopf algebra $\Bbbk\mathcal{T}$ is semisimple, since the underlying algebra is a groupoid algebra.

(ii). The map $\lambda : \mathcal{B} \to \Bbbk^{\times}$, $\lambda(A) = \dfrac{\ulcorner(A)}{\llcorner(A)}$ is a character with respect to vertical composition, see Lemma 3.11.

Theorem 2.3 will be proved in Section 3. We find appropriate to make the following observation. The algebra and coalgebra structures depend intrinsically on the 'double' nature of operations in $\mathcal{T}$. It would seem that the construction of $\Bbbk\mathcal{T}$ shows some preference for the vertical composition over the horizontal one. However, as the following remark shows, there is no such preference: the symmetry of the construction is hidden behind the choice of basis in $\Bbbk\mathcal{T}$.

Remark 2.5. Let H be the vector space with basis $\{\underline{A}\}_{A\in\mathcal{B}}$, and multiplication and comultiplication defined by

$$\underline{A}.\underline{B} = \begin{cases} \dfrac{1}{\urcorner(A)}\, \underline{C} = \dfrac{1}{\lrcorner(B)}\, \underline{C}, & \text{if } \dfrac{A}{B}, \quad \text{where } C = \dfrac{A}{B}; \\ 0, & \text{otherwise}, \end{cases} \tag{2.14}$$

$$\Delta(\underline{A}) = \sum_{XY=A} \underline{X} \otimes \underline{Y}, \tag{2.15}$$

for all $A, B \in \mathcal{B}$. Then the linear isomorphism $H \to \Bbbk\mathcal{T}$, $\underline{A} \mapsto \dfrac{1}{\urcorner(A)} A$, preserves multiplication and comultiplication, in view of Proposition 1.19 (ii) and (i), respectively.

2.3. **The square of the antipode.**

In this subsection we give the relations between corner functions and the square of the antipode. The proof of the following lemma is straightforward.

Lemma 2.6. *Let $A \in \mathcal{B}$. Then* $\mathcal{S}^2(A) = \frac{\ulcorner(A)\lrcorner(A)}{\llcorner(A)\urcorner(A)} A.$ □

It follows from Proposition 1.17, that if $tl(A) = bl(A)$ and $br(A) = tr(A)$, then $\mathcal{S}^2(A) = A$.

Recall from [N2, Definition 2.2.2] that a weak Hopf algebra is called *regular* if $\mathcal{S}^2 = \mathrm{id}$ on the source and target subalgebras. Actually, if $\mathcal{S}^2 = \mathrm{id}$ on the source (respectively, target) subalgebra, then also $\mathcal{S}^2 = \mathrm{id}$ on the target (respectively, source) subalgebra.

Proposition 2.7. *$\Bbbk\mathcal{T}$ is a regular weak Hopf algebra.*

Proof. Let $D \in \mathbf{D}$ and $E \in \mathbf{E}$. Then

$$\mathcal{S}(\mathbf{1}_E) = {}_{E^{-1}}\mathbf{1}, \qquad \mathcal{S}({}_D\mathbf{1}) = \frac{\theta(s(D))}{\theta(e(D))}\mathbf{1}_{D^{-1}} = \mathbf{1}_{D^{-1}},$$

the last equality by Lemma 1.18. Hence $\mathcal{S}^2 = \mathrm{id}$ on the source subalgebra. This proves the proposition. □

It is natural to ask whether $\mathcal{S}^2 = \mathrm{id}$ on the whole weak Hopf algebra $\Bbbk\mathcal{T}$. It is possible to give a positive answer in some cases, *e. g.* if $\mathcal{T}$ is a double group (that is, if $\mathcal{P}$ has just one element), if $\mathcal{T}$ is vacant, or if $\mathcal{T}$ is locally trivial– see Example 2.9 below. But there are double groupoids $\mathcal{T}$ with $\mathcal{S}^2 \neq \mathrm{id}$.

Example 2.8. We show that, for any positive rational number r, there exists a double groupoid $\mathcal{T}$ satisfying (2.10) and a box $A \in \mathcal{B}$ such that $\mathcal{S}^2(A) = rA$.

Let m, n be natural numbers. Let $\mathcal{P}$ be the set with $m + n + 3$ elements labeled

$$P,\ Q,\ R,\ S_1, \dots,\ S_m,\ T_1, \dots,\ T_n.$$

Let $\mathcal{H}$ and $\mathcal{V}$ be the groupoids with base $\mathcal{P}$ corresponding to the equivalence relations $\sim_{\mathcal{H}}$ and $\sim_{\mathcal{V}}$ given, respectively, by the following partitions:

$$\begin{aligned} \sim_{\mathcal{H}}\colon &\quad \{P,\ Q,\ T_1, \dots,\ T_n\} \bigcup \{R,\ S_1, \dots,\ S_m\}, \\ \sim_{\mathcal{V}}\colon &\quad \{P, R\} \bigcup \{Q,\ S_1, \dots,\ S_m,\ T_1, \dots,\ T_n\}. \end{aligned}$$

Let $\mathcal{T} = \begin{matrix} \mathcal{B} & \rightrightarrows & \mathcal{H} \\ \downdownarrows & & \downdownarrows \\ \mathcal{V} & \rightrightarrows & \mathcal{P} \end{matrix}$ be the double groupoid consisting of commuting squares in the coarse groupoid $\mathcal{P} \times \mathcal{P}$, with horizontal arrows in $\mathcal{H}$ and vertical arrows in $\mathcal{V}$. We claim that $\mathcal{T}$ satisfies the filling condition (2.10).

Moreover, any $u \in \mathcal{P} \times \mathcal{P}$ can be expressed as a product gx with $g \in \mathcal{V}$, $x \in \mathcal{H}$. Indeed, if $u \in \mathcal{H}$ or $\mathcal{V}$, this is clear. The remaining possibilities are:

$$\begin{aligned}
u &= (P, S_j) = (P, R)(R, S_j),\\
u &= (Q, R) = (Q, S_j)(S_j, R),\\
u &= (S_j, P) = (S_j, Q)(Q, P),\\
u &= (R, Q) = (R, P)(P, Q),\\
u &= (R, T_i) = (R, P)(P, T_i),\\
u &= (T_i, R) = (T_i, S_j)(S_j, R).
\end{aligned}$$

Next, if $X \in \mathcal{P}$ then $\theta(X)$ equals the number of arrows in $\mathcal{H} \cap \mathcal{V}$ with source X. In the present example, this number equals

$$\#\{Y \in \mathcal{P} : Y \sim_{\mathcal{V}} X, Y \sim_{\mathcal{H}} X\}.$$

Thus

$$\theta(P) = 1 = \theta(R), \qquad \theta(Q) = n + 1 = \theta(T_i), \qquad \theta(S_j) = m,$$

$1 \le j \le m$, $1 \le i \le n$. Consider now the box

$$A = \begin{array}{c} P \quad Q \\ \square \\ R \quad S_1 \end{array} \in \mathcal{B}.$$

Then

$$\mathcal{S}^2(A) = \frac{\ulcorner(A)\lrcorner(A)}{\llcorner(A)\urcorner(A)} A = \frac{\theta(S_1)\theta(P)}{\theta(Q)\theta(R)} A = \frac{m}{n+1} A.$$

Example 2.9. Let us first recall the following definitions [BMa, Definition 2.3]. Let $\mathcal{T}$ be a finite double groupoid.

(a) $\mathcal{T}$ is *horizontally transitive* if every configuration of matching sides

$$\begin{array}{c} l \sqcup r \\ b \end{array},$$

$l, r \in \mathcal{V}$, $b \in \mathcal{H}$, can be completed to at least one box in $\mathcal{B}$. Equivalently, with $\sqcap$ instead of $\sqcup$.

(b) $\mathcal{T}$ is *vertically transitive* if every configuration of matching sides

$$\sqsupset$$

can be completed to at least one box in $\mathcal{B}$. Equivalently, with $\sqsubset$ instead of $\sqsupset$.

(c) $\mathcal{T}$ is *transitive* or *locally trivial* if it is both vertically and horizontally transitive.

Note that any double groupoid that is either horizontally or vertically transitive satisfies the filling condition (2.10). Indeed, assume that $\mathcal{T}$ is vertically transitive and fix $(x, g) \in \mathcal{H}\,{}_r\!\times_t \mathcal{V}$. Then the configuration

$$\begin{array}{ccc} & x & \\ & \rule{2em}{0.4pt} & \Big| \; g \\ & \operatorname{id} b(g) & \end{array}$$

can be completed to at least one box in $\mathcal{B}$.

Lemma 2.10. *(i). If $\mathcal{T}$ is vertically transitive then the corresponding corner function θ is constant along the connected components of $\mathcal{H} \rightrightarrows \mathcal{P}$. Hence $\mathcal{S}^2 = \operatorname{id}$ on $\Bbbk\mathcal{T}$.*

(ii). If $\mathcal{T}$ is horizontally transitive then the corresponding θ is constant along the connected components of $\mathcal{V} \rightrightarrows \mathcal{P}$. Hence $\mathcal{S}^2 = \operatorname{id}$ on $\Bbbk\mathcal{T}$.

Proof. (i). Let $P, Q \in \mathcal{P}$ and $x \in \mathcal{H}$ such that $P = l(x)$, $Q = r(x)$. Complete the configuration of matched sides

$$\begin{array}{ccc} & x & \\ \operatorname{id} P \; \Big\| & \rule{2em}{0.4pt} & \\ & \operatorname{id} P & \end{array}$$

to a box $D \in \mathbf{D}$. Then D connects P and Q, so $\theta(P) = \theta(Q)$ by Lemma 1.18. The second claim follows from Lemma 2.6.

The proof of (ii) is analogous, or else follows from (i) by passing to the transpose double groupoid. $\square$

2.4. **The category** $\operatorname{Rep} \Bbbk\mathcal{T}$.

Let $\mathcal{T}$ be a finite double groupoid satisfying the filling condition (2.10). Let $\Bbbk\mathcal{T}$ be the quantum groupoid associated to $\mathcal{T}$ as in Theorem 2.3. Consider the category $\mathcal{C} := \operatorname{Rep} \Bbbk\mathcal{T}$ of finite dimensional representations of $\Bbbk\mathcal{T}$. This is a semisimple category, with a finite number of simple objects.

Objects of $\mathcal{C}$ are finite dimensional $\Bbbk$-linear bundles over the vertical groupoid $\mathcal{B} \rightrightarrows \mathcal{H}$; that is, $\mathcal{H}$-graded vector spaces endowed with a left action of the vertical groupoid $\mathcal{B} \rightrightarrows \mathcal{H}$ by linear isomorphisms, *cf.* [AN]. Here, a box $A \in \mathcal{B}$ acts on $V = \oplus_{x \in \mathcal{H}} V_x$ via the linear isomorphism $A : V_{b(A)} \to V_{t(A)}$.

There is a structure of rigid monoidal category on $\operatorname{Rep}\Bbbk\mathcal{T}$. The unit object is the target subalgebra $\Bbbk\mathcal{T}_t = \Bbbk(\mathbf{1}_E : E \in \mathbf{E}) \simeq (\Bbbk\mathbf{E})^{\mathrm{op}}$. If V, W are $\mathcal{T}$-bundles then $V \otimes W := V \otimes_{\mathbf{E}^{\mathrm{op}}} W$, where $\Bbbk\mathbf{E}^{\mathrm{op}}$ acts on the left via the target subalgebra and on the right via the left action of the source subalgebra. The action of $\mathcal{B}$ on $V \otimes W$ is given by Δ. The *dual* V^* of an object $V = \oplus_{x\in\mathcal{H}} V_x \in \mathcal{C}$ has $\mathcal{H}$-grading $(V^*)_x = (V_{x^{-1}})^*$, $x \in \mathcal{H}$. If $A \in \mathcal{B}$, then it acts by the transpose of $A^{-1} : (V^*)_{b(A)} \to (V^*)_{t(A)}$.

The core groupoids $\mathbf{D}$ and $\mathbf{E}$ are embedded "diagonally" into the algebra $\Bbbk\mathcal{T}$. The multiplication between elements of $\mathbf{D}$ or $\mathbf{E}$ and those in $\mathcal{B}$ is related to the actions in Proposition 1.8.

Lemma 2.11. *Let $A \in \mathcal{B}$, $D \in \mathbf{D}$, $E \in \mathbf{E}$. Then*

$$ {}_D\mathbf{1}.A = \begin{cases} D \rightharpoonup A, & \textit{if } rt(A) = e(D), \\ 0, & \textit{otherwise;} \end{cases} \tag{2.16}$$

$$ A.{}_D\mathbf{1} = \begin{cases} A \leftharpoonup D, & \textit{if } rb(A) = s(D), \\ 0, & \textit{otherwise;} \end{cases} \tag{2.17}$$

$$ \mathbf{1}_E.A = \begin{cases} A \leftharpoondown E, & \textit{if } lt(A) = s(E), \\ 0, & \textit{otherwise;} \end{cases} \tag{2.18}$$

$$ A.\mathbf{1}_E = \begin{cases} E \rightharpoondown A, & \textit{if } lb(A) = e(E), \\ 0, & \textit{otherwise.} \end{cases} \tag{2.19}$$

$\square$

Let $D \in \mathbf{D}$, $E \in \mathbf{E}$, $z, w \in \mathcal{H}$. Then, by the Lemma above, we have

$$ {}_D\mathbf{1}.\mathbf{id}\, z = \begin{cases} \{\mathbf{id}\, zD\} = \mathbf{id}\,(zt(D)).{}_D\mathbf{1}, & \text{if } r(z) = e(D), \\ 0, & \text{otherwise;} \end{cases} \tag{2.20}$$

$$ \mathbf{1}_E.\mathbf{id}\, w = \begin{cases} \{E\mathbf{id}\, b(E)^{-1}w\} = \mathbf{id}\,(b(E)^{-1}w).{}_E\mathbf{1}, & \text{if } l(w) = e(E), \\ 0, & \text{otherwise.} \end{cases} \tag{2.21}$$

If $D \in \mathbf{D}$, set $\theta(D) := \theta(e(D)) = \theta(s(D))$, *cf.* Lemma 1.18. If $E \in \mathbf{E}$, set $\theta(E) := \theta(E^{-1})$.

Proposition 2.12.

$$\Delta(1) = \sum_{D\in\mathbf{D}} \frac{1}{\theta(D)}\, {}_D\mathbf{1} \otimes \mathbf{1}_{D^\dagger} \tag{2.22}$$

$$= \sum_{E\in\mathbf{E}} \frac{1}{\theta(E)}\, {}_{E^\dagger}\mathbf{1} \otimes \mathbf{1}_{E}; \tag{2.23}$$

$$\Delta(\mathbf{id}\,x) = \Delta(1) \sum_{zw=x} \mathbf{id}\,z \otimes \mathbf{id}\,w, \qquad x\in\mathcal{H}. \tag{2.24}$$

Proof. Let $x\in\mathcal{H}$. We compute

$$\begin{aligned}
\Delta(\mathbf{id}\,x) &= \sum_{AB=\mathbf{id}\,x} \frac{1}{\urcorner(B)}\, A\otimes B \\
&= \sum_{D\in\mathbf{D},\, z\in\mathcal{H}: r(z)=e(D)} \quad \sum_{y\in\mathcal{H},\, l(y)=e(E); \{\mathbf{id} z\, DE\,\mathbf{id} y\}=\mathbf{id} x} \frac{1}{\theta(D)} \{\mathbf{id} z\, D\} \otimes \{E\,\mathbf{id} y\} \\
&= \sum_{D\in\mathbf{D},\, z\in\mathcal{H}: r(z)=e(D)} \frac{1}{\theta(D)} \{\mathbf{id} z\, D\} \otimes \{D^\dagger \mathbf{id}(t(D)^{-1} z^{-1} x)\}.
\end{aligned}$$

Here the first equality is by definition; the second uses the bijections (1.12) and (1.13); the third, Remark 1.6. This implies (2.22), and (2.23) follows by a routine change of variables. Starting from the last equation and using (2.20), we have

$$\Delta(\mathbf{id}\,x) = \sum_{D\in\mathbf{D},\, z\in\mathcal{H}: r(z)=e(D)} \frac{1}{\theta(D)}\, {}_D\mathbf{1}.\mathbf{id}\,z \otimes \mathbf{1}_{D^\dagger}.\mathbf{id}(z^{-1}x),$$

since $t(D) = b(D^\dagger)$. This implies (2.24). □

As a consequence, if V, W are $\mathcal{T}$-bundles then the homogeneous components of the tensor product are given by

$$(V\otimes W)_x = \Delta(1)\Big(\sum_{zw=x} V_z \otimes W_w\Big), \qquad x\in\mathcal{H}.$$

3. Cocycle deformations

3.1. Generalities.

Let $\mathcal{T}$ be a *finite* double groupoid. As before, let $\Bbbk\mathcal{T}$ denote the $\Bbbk$-vector space with basis $\mathcal{B}$. We shall consider the following structures on $\Bbbk\mathcal{T}$.

Algebra structure.

We deform the groupoid algebra structure on $\Bbbk\mathcal{T}$ corresponding to the vertical composition groupoid $\mathcal{B} \rightrightarrows \mathcal{H}$.

Lemma 3.1. *Let $\sigma : \mathcal{B}\,{}_b\!\times_t \mathcal{B} \to \Bbbk^\times$ be a function and define a multiplication in $\Bbbk\mathcal{T}$ by*

$$A.B = \begin{cases} \sigma(A,B)\, \dfrac{A}{B}, & \text{if } \dfrac{A}{B}, \\ 0, & \text{otherwise.} \end{cases} \qquad \text{for all } A, B \in \mathcal{B}. \tag{3.1}$$

This multiplication is associative if and only if σ is a vertical 2-cocycle:

$$\sigma(A,B)\sigma\left(\frac{A}{B}, C\right) = \sigma(B,C)\sigma\left(A, \frac{B}{C}\right), \tag{3.2}$$

for all $A, B, C \in \mathcal{B}$: $\dfrac{\dfrac{A}{B}}{C}$. If this happens, there is a unit

$$\mathbf{1}^\sigma := \sum_{x\in\mathcal{H}} \frac{1}{\sigma(\mathbf{id}\,x, \mathbf{id}\,x)}\,\mathbf{id}\,x.$$

Proof. The proof of the first claim is routine. For the second, the identities $\sigma(A, \mathbf{id}b(A)) = \sigma(\mathbf{id}b(A), \mathbf{id}b(A))$, $\sigma(\mathbf{id}t(A), A) = \sigma(\mathbf{id}t(A), \mathbf{id}t(A))$ are needed, but these follow from (3.2). $\square$

If (3.2) holds, the unit is $\mathbf{1} := \sum_{x\in\mathcal{H}} \mathbf{id}\,x$ if and only if σ is normalized:

$$\sigma(A, \mathrm{id}\ t(A)) = \sigma(\mathrm{id}\ b(A), A) = 1, \text{ for all } A \in \mathcal{B}. \tag{3.3}$$

Up to a change of basis, one can always assume that σ is normalized.

Coalgebra structure.

Dually, we shall deform the groupoid coalgebra structure on $\Bbbk\mathcal{T}$ corresponding to the horizontal composition groupoid $\mathcal{B} \rightrightarrows \mathcal{V}$.

Lemma 3.2. *Let $\tau : \mathcal{B}\,{}_r\!\times_l \mathcal{B} \to \Bbbk^\times$ be a function and define a comultiplication in $\Bbbk\mathcal{T}$ by*

$$\Delta(A) = \sum_{A=BC} \tau(B,C)\, B \otimes C, \qquad A \in \mathcal{B}. \tag{3.4}$$

This comultiplication is coassociative if and only if

$$\tau(A,B)\tau(AB,C) = \tau(B,C)\tau(A,BC), \tag{3.5}$$

for all $A, B, C \in \mathcal{B}$: $A|B|C$. If this happens, then there is a counit $\varepsilon^\tau : \Bbbk\mathcal{T} \to \Bbbk$ given by

$$\varepsilon^\tau(A) = \begin{cases} \dfrac{1}{\tau(A,A)}, & \text{if } A \in \mathcal{V}, \\ 0, & \text{otherwise.} \end{cases}$$

$\square$

Weak Hopf algebra structure.

We fix $\sigma : \mathcal{B}\,{}_b\!\times_t \mathcal{B} \to \Bbbk^\times$ satisfying (3.2) and (3.3), and $\tau : \mathcal{B}\,{}_r\!\times_l \mathcal{B} \to \Bbbk^\times$ satisfying (3.5). We denote by $\Bbbk_\sigma^\tau\mathcal{T}$ the vector space $\Bbbk\mathcal{T}$ with multiplication (3.1) and comultiplication (3.4). Observe that it is not possible to normalize σ and τ simultaneously. We begin by the following straightforward proposition.

Proposition 3.3. *$\Bbbk_\sigma^\tau\mathcal{T}$ is a weak bialgebra if and only if the following hold:*

$$\sigma(A,B)\tau(X,Y) = \sum \sigma(U,R)\sigma(V,S)\tau(U,V)\tau(R,S), \tag{3.6}$$

for all $A,B,X,Y \in \mathcal{B} : XY = \genfrac{}{}{0pt}{}{A}{B}$, where the index set is $[X,Y,A,B]$;

$$\tau\left(A, \genfrac{}{}{0pt}{}{U}{V}\right)\tau\left(A\genfrac{}{}{0pt}{}{U}{V}, C\right) = \tau(A,U)\tau(V,C)\sigma(U,V), \tag{3.7}$$

$$\tau\left(A, \genfrac{}{}{0pt}{}{W}{Z}\right)\tau\left(A\genfrac{}{}{0pt}{}{W}{Z}, C\right) = \tau(A,Z)\tau(W,C)\sigma(W,Z), \tag{3.8}$$

for all $A,C,U,V,W,Z \in \mathcal{B}:\ AU, VC, AZ, WC \in \mathcal{H}$;

(3.9)
$$\sigma(A,UV)\sigma\left(\genfrac{}{}{0pt}{}{A}{UV}, C\right)\tau\left(\genfrac{}{}{0pt}{}{A}{U}, \genfrac{}{}{0pt}{}{A}{U}\right)\tau\left(\genfrac{}{}{0pt}{}{V}{C}, \genfrac{}{}{0pt}{}{V}{C}\right) = \tau(U,V)\sigma(A,U)\sigma(V,C)\tau\left(\genfrac{}{}{0pt}{}{A}{\genfrac{}{}{0pt}{}{UV}{C}}, \genfrac{}{}{0pt}{}{A}{\genfrac{}{}{0pt}{}{UV}{C}}\right),$$

$$\begin{aligned}\sigma(A,WZ)\sigma\left(\genfrac{}{}{0pt}{}{A}{WZ}, C\right)\tau\left(\genfrac{}{}{0pt}{}{A}{Z}, \genfrac{}{}{0pt}{}{A}{Z}\right)\tau\left(\genfrac{}{}{0pt}{}{W}{C}, \genfrac{}{}{0pt}{}{W}{C}\right)& \\ = \tau(W,Z)\sigma(A,Z)\sigma(W,C)\tau\left(\genfrac{}{}{0pt}{}{A}{\genfrac{}{}{0pt}{}{WZ}{C}}, \genfrac{}{}{0pt}{}{A}{\genfrac{}{}{0pt}{}{WZ}{C}}\right)&,\end{aligned} \tag{3.10}$$

for all $A,C,U,V,W,Z \in \mathcal{B} : \genfrac{}{}{0pt}{}{A}{U}, \genfrac{}{}{0pt}{}{V}{C}, \genfrac{}{}{0pt}{}{A}{Z}, \genfrac{}{}{0pt}{}{W}{C} \in \mathcal{V}$.

In this case, the source and target maps are given, respectively, by

(3.11)
$$\epsilon_s(A) = \begin{cases} \sum_{x\in\mathcal{H},\, r(x)=br(A)} \dfrac{\tau(\mathbf{id}x\,\phi(A), \phi(A)^h)\sigma(A,\phi(A)^h)}{\tau(\mathbf{id}r(A), \mathbf{id}r(A))}\{\mathbf{id}x\,\phi(A)\}, & \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \textit{if} \quad t(A)\in\mathcal{P}, & \\ 0, \qquad \textit{otherwise}; & \end{cases}$$

(3.12)
$$\epsilon_t(A) = \begin{cases} \sum_{y\in\mathcal{H},\, l(y)=bl(A)} \dfrac{\tau(\psi(A)^h, \psi(A)\mathbf{id}y)\sigma(\psi(A)^h, A)}{\tau(\mathbf{id}l(A), \mathbf{id}l(A))}\{\psi(A)\mathbf{id}y\}, & \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \textit{if} \quad b(A)\in\mathcal{P}, & \\ 0, \qquad \textit{otherwise}. & \end{cases}$$

Proof. We first show:

Step 1. The comultiplication (3.4) is multiplicative with respect to the multiplication (3.1) if and only if (3.6) holds.

Let $A, B \in \mathcal{B}$. It follows from the definitions that

$$\Delta(A.B) = \sum_{XY = \genfrac{}{}{0pt}{}{A}{B}} \sigma(A,B)\tau(X,Y) \quad X \otimes Y.$$

On the other hand,

$$\Delta(A).\Delta(B) = \sum \sigma(U,R)\sigma(V,S)\tau(U,V)\tau(R,S) \genfrac{}{}{0pt}{}{U}{R} \otimes \genfrac{}{}{0pt}{}{V}{S},$$

where the sum runs over all elements $U, V, R, S \in \mathcal{B}$, such that $\begin{array}{c|c} U & V \\ \hline R & S \end{array}$, $UV = A$ and $RS = B$. It is thus clear that $\Delta(A).\Delta(B) = 0 = \Delta(A.B)$, if A and B are not vertically composable. If $\frac{A}{B}$, then (2.1) is equivalent to (3.6).

Step 2. The first, resp. the second, equality in (2.2) is equivalent to (3.7), resp. (3.8).

We have

$$\Delta^{(2)}(\mathbf{1}) = \sum_{ABC \in \mathcal{H}} \tau(A,B)\,\tau(AB,C)\ A \otimes B \otimes C.$$

On the other hand, by (3.3), we have

$$\begin{aligned}
&(\Delta(\mathbf{1}) \otimes \mathbf{1})\,(\mathbf{1} \otimes \Delta(\mathbf{1})) \\
&\qquad = \sum_{\genfrac{}{}{0pt}{}{U}{V} = B,\, AU \in \mathcal{H},\, VC \in \mathcal{H}} \tau(A,U)\tau(V,C)\sigma(U,V) \quad A \otimes \genfrac{}{}{0pt}{}{U}{V} \otimes C \\
&(\mathbf{1} \otimes \Delta(\mathbf{1}))\,(\Delta(\mathbf{1}) \otimes \mathbf{1}) \\
&\qquad = \sum_{\genfrac{}{}{0pt}{}{W}{Z} = B,\, AZ \in \mathcal{H},\, WC \in \mathcal{H}} \tau(A,Z)\tau(W,C)\sigma(W,Z) \quad A \otimes \genfrac{}{}{0pt}{}{W}{Z} \otimes C.
\end{aligned}$$

Thanks to the equivalences (i) $\iff$ (ii) and (i) $\iff$ (iii) in Lemma 1.1, the claim in Step 2 follows.

Step 3. The first, resp. the second, equality in (2.3) is equivalent to (3.9), resp. (3.10).

Let $A, B, C \in \mathcal{B}$. We have

$$\epsilon(A.B.C) = \begin{cases} \sigma(A,B)\sigma\left(\begin{smallmatrix} A \\ B\end{smallmatrix}, C\right)\tau\left(\begin{smallmatrix} A & A \\ B, & B \\ C & C\end{smallmatrix}\right)^{-1}, & \text{if } \begin{smallmatrix} A \\ B \\ C\end{smallmatrix} \in \mathcal{V}, \\ 0, \quad \text{otherwise.} \end{cases}$$

On the other hand, we have

$$\begin{aligned} \epsilon(A.B_1)\epsilon(B_2.C) &= \sum_{UV=B} \tau(U,V)\,\epsilon(A.U)\epsilon(V.C) \\ &= \sum_{UV=B,\, \begin{smallmatrix} A \\ U\end{smallmatrix}\in\mathcal{V},\, \begin{smallmatrix} V \\ C\end{smallmatrix}\in\mathcal{V}} \tau(U,V)\sigma(A,U)\sigma(V,C)\tau\left(\begin{smallmatrix} A & A \\ U, & U\end{smallmatrix}\right)^{-1}\tau\left(\begin{smallmatrix} V & V \\ C, & C\end{smallmatrix}\right)^{-1}. \end{aligned}$$

By Lemma 1.2, there exist unique such U and V in the index set of the last sum. Thus the first equality in (2.3) is equivalent to (3.9). Similarly, the second equality in (2.3) is equivalent to (3.10).

Step 4. The source and target maps.

Let $A \in \mathcal{B}$. We compute

(3.13)
$$\epsilon_s(A) = \sum_{XY\in\mathcal{H}} \tau(X,Y)\ X\ \epsilon(A.Y) = \sum_{XY\in\mathcal{H},\, \begin{smallmatrix} A \\ Y\end{smallmatrix}\in\mathcal{V}} \frac{\tau(X,Y)\sigma(A,Y)}{\tau\left(\begin{smallmatrix} A \\ Y\end{smallmatrix}, \begin{smallmatrix} A \\ Y\end{smallmatrix}\right)} \quad X.$$

By Lemma 1.11 (i), $Y = \phi(A)^h$, $X = \mathbf{id} x\phi(A)$, for a unique $x \in \mathcal{H}$ such that $r(x) = rb(A)$, whence formula (3.11).

As for the target map, we have

$$\epsilon_t(A) = \sum_{XY\in\mathcal{H}} \tau(X,Y)\ \epsilon(X.A)Y = \sum_{XY\in\mathcal{H},\, \begin{smallmatrix} X \\ A\end{smallmatrix}\in\mathcal{V}} \frac{\tau(X,Y)\sigma(X,A)}{\tau\left(\begin{smallmatrix} X \\ A\end{smallmatrix}, \begin{smallmatrix} X \\ A\end{smallmatrix}\right)}\, Y. \tag{3.14}$$

By Lemma 1.11 (ii), (3.12) follows. □

We denote by $\Bbbk^\tau\mathcal{T}$ the algebra and coalgebra $\Bbbk^\tau_\sigma\mathcal{T}$ when $\sigma = 1$ is the trivial cocycle.

Corollary 3.4. *$\Bbbk^\tau\mathcal{T}$ is a weak bialgebra if and only if*

$$\tau(X,Y) = \sum \tau(U,V)\tau(R,S), \tag{3.15}$$

for all $A, B, X, Y \in \mathcal{B} : XY = \begin{smallmatrix} A \\ B \end{smallmatrix}$, *where the index set is* $[X, Y, A, B]$*;*

$$\tau\left(A, \begin{smallmatrix} U \\ V \end{smallmatrix}\right) \tau\left(A \begin{smallmatrix} U \\ V \end{smallmatrix}, C\right) = \tau(A, U)\tau(V, C), \tag{3.16}$$

$$\tau\left(A, \begin{smallmatrix} W \\ Z \end{smallmatrix}\right) \tau\left(A \begin{smallmatrix} W \\ Z \end{smallmatrix}, C\right) = \tau(A, Z)\tau(W, C), \tag{3.17}$$

for all $A, C, U, V, W, Z \in \mathcal{B} : \; AU, VC, AZ, WC \in \mathcal{H}$*;*

$$\tau\left(\begin{smallmatrix} A \\ U \end{smallmatrix}, \begin{smallmatrix} A \\ U \end{smallmatrix}\right) \tau\left(\begin{smallmatrix} V \\ C \end{smallmatrix}, \begin{smallmatrix} V \\ C \end{smallmatrix}\right) = \tau(U, V)\tau\left(\begin{smallmatrix} A \\ B \\ C \end{smallmatrix}, \begin{smallmatrix} A \\ B \\ C \end{smallmatrix}\right), \tag{3.18}$$

$$\tau\left(\begin{smallmatrix} A \\ Z \end{smallmatrix}, \begin{smallmatrix} A \\ Z \end{smallmatrix}\right) \tau\left(\begin{smallmatrix} W \\ C \end{smallmatrix}, \begin{smallmatrix} W \\ C \end{smallmatrix}\right) = \tau(W, Z)\tau\left(\begin{smallmatrix} A \\ B \\ C \end{smallmatrix}, \begin{smallmatrix} A \\ B \\ C \end{smallmatrix}\right), \tag{3.19}$$

for all $A, C, U, V, W, Z \in \mathcal{B} : \begin{smallmatrix} A \\ U \end{smallmatrix}, \begin{smallmatrix} V \\ C \end{smallmatrix}, \begin{smallmatrix} A \\ Z \end{smallmatrix}, \begin{smallmatrix} W \\ C \end{smallmatrix} \in \mathcal{V}$*.* $\square$

3.2. **Generalized corner functions.**

We consider now the following particular case. We fix a function $\vartheta : \mathcal{P} \to \Bbbk^{\times}$ and introduce $\urcorner : \mathcal{B} \to \Bbbk^{\times}$, $\tau : \mathcal{B}\,{}_r\!\times_l \mathcal{B} \to \Bbbk^{\times}$ by

$$\urcorner(B) = \vartheta(bl(B)), \qquad \tau(A, B) = \urcorner(B).$$

It is immediate that

$$\urcorner(XY) = \urcorner(X), \tag{3.20}$$

$$\urcorner\left(\begin{smallmatrix} U \\ V \end{smallmatrix}\right) = \urcorner(V), \tag{3.21}$$

for all $X, Y, U, V \in \mathcal{B}$ appropriately composable. Note that condition (3.5) for τ is equivalent to condition (3.20) for $\urcorner$. Also, condition (3.17) for τ is equivalent to condition (3.21) for $\urcorner$.

Remark 3.5. Let $\urcorner : \mathcal{B} \to \Bbbk^{\times}$ be any function and define $\tau : \mathcal{B}\,{}_r\!\times_l \mathcal{B} \to \Bbbk^{\times}$ by $\tau(A, B) = \urcorner(B)$. Suppose that $\urcorner$ satisfies (3.20) and (3.21). Then $\urcorner(B) = \vartheta(bl(B))$ for any $B \in \mathcal{B}$, where $\vartheta : \mathcal{P} \to \Bbbk^{\times}$ is given by $\vartheta(P) = \urcorner(\Theta_P)$, $P \in \mathcal{P}$.

To stress the dependence on ϑ, we shall use the notation $\Bbbk^{(\vartheta)}\mathcal{T}$ instead of $\Bbbk^{\tau}\mathcal{T}$.

Theorem 3.6. $\Bbbk^{(\vartheta)}\mathcal{T}$ *is a weak bialgebra if and only if*

$$1 = \sum_{V \in \mathcal{B}: t(V) = x,\, r(V) = g} \urcorner(V), \tag{3.22}$$

for all $(g, x) \in \mathcal{V}\,{}_t\!\times_r \mathcal{H}$*.*

When this condition holds, $\Bbbk^{(\vartheta)}\mathcal{T}$ is a semisimple weak Hopf algebra, with antipode determined by the formula

$$\mathcal{S}(A) = \frac{\daleth(A^{-1})}{\daleth(A^h)}\, A^{-1} = \frac{\vartheta(tr(A))}{\vartheta(br(A))}\, A^{-1}, \tag{3.23}$$

for all $A \in \mathcal{B}$. The source and target maps are given, respectively, by

$$\epsilon_s(A) = \begin{cases} {}_{\phi(A)}\mathbf{1}, & \text{if} \quad t(A) \in \mathcal{P}, \\ 0, & \text{otherwise}; \end{cases} \tag{3.24}$$

$$\epsilon_t(A) = \begin{cases} \dfrac{\daleth(A^{-1})}{\daleth(A^h)}\, \mathbf{1}_{\psi(A)} & \text{if} \quad b(A) \in \mathcal{P}, \\ 0, & \text{otherwise}. \end{cases} \tag{3.25}$$

The source and target subalgebras are isomorphic to the groupoid algebras $\Bbbk\mathbf{D}$ and $(\Bbbk\mathbf{E})^{\mathrm{op}}$, respectively.

Note that (3.22) implies that $\mathcal{T}$ satisfies the filling condition (2.10).

Proof. Clearly, (3.15) is equivalent to

$$\daleth(Y) = \tau(X, Y) = \sum \tau(U, V)\tau(R, S) = \sum \daleth(V)\daleth(S)$$

for all $A, B, X, Y \in \mathcal{B} : XY = \dfrac{A}{B}$, where the index set is $[X, Y, A, B]$. Since $\daleth(Y) = \daleth(S)$, we see that (3.22) is equivalent to (3.15).

Now conditions (3.17) and (3.19) follow at once from (3.21), while (3.16) and (3.18) follow easily using that $VC \in \mathcal{H}$, resp. that $\frac{A}{U} \in \mathcal{V}$.

It remains to consider the axioms of the antipode. We first show (2.4). Let $A \in \mathcal{B}$. Then

$$m(\mathrm{id} \otimes \mathcal{S})\Delta(A) = \sum_{\frac{U}{V^{-1}},\, UV = A} \tau(U, V) \frac{\daleth(V^{-1})}{\daleth(V^h)}\; \frac{U}{V^{-1}};$$

thus this sum vanishes, and so equation (2.4) is true, unless $b(A) \in \mathcal{P}$. We shall now assume that this is the case. Letting $Y = \dfrac{U}{V^{-1}}$, and $X =$

$\begin{Bmatrix}\mathbf{id}\, l(A)\\ A^v\end{Bmatrix}$, the last sum equals

$$\begin{aligned}
&\sum_{XY\in\mathcal{H},\, \genfrac{}{}{0pt}{}{X}{A}\in\mathcal{V}} \quad \sum_{UV=A,\, \genfrac{}{}{0pt}{}{U}{V^{-1}}=Y} \daleth(V)\frac{\daleth(V^{-1})}{\daleth(V^h)}\, Y\\
&= \sum_{XY\in\mathcal{H},\, \genfrac{}{}{0pt}{}{X}{A}\in\mathcal{V}} \frac{\daleth(Y)}{\daleth(A)} \sum_{UV=A,\, \genfrac{}{}{0pt}{}{U}{V^{-1}}=Y} \daleth(V)\, Y\\
&= \sum_{XY\in\mathcal{H},\, \genfrac{}{}{0pt}{}{X}{A}\in\mathcal{V}} \frac{\daleth(Y)}{\daleth(A)} Y\\
&= \sum_{XY\in\mathcal{H},\, \genfrac{}{}{0pt}{}{X}{A}\in\mathcal{V}} \frac{\tau(X,Y)}{\tau(\genfrac{}{}{0pt}{}{X}{A},\genfrac{}{}{0pt}{}{X}{A})}\, Y,
\end{aligned}$$

and we are done by (3.14). We have used that $\daleth(Y)=\daleth(V^{-1})$ and $\daleth(V^h)=\vartheta(br(V))=\vartheta(tl(V^{-1}))=\vartheta(bl(U))=\daleth(A)$.

Relation (2.5) is proved similarly: let $A\in\mathcal{B}$. Then

$$m(\mathcal{S}\otimes\mathrm{id})\Delta(A) = \sum_{\frac{U^{-1}}{V},\, UV=A} \daleth(V)\frac{\daleth(U^{-1})}{\daleth(U^h)} \qquad \genfrac{}{}{0pt}{}{U^{-1}}{V};$$

as before, this sum vanishes, implying equation (2.5), in the case where $t(A)$ is not in $\mathcal{P}$. We shall assume that $t(A)\in\mathcal{P}$. Letting $X=\genfrac{}{}{0pt}{}{U^{-1}}{V}$ and $Y=\begin{Bmatrix}A^v\\ \mathbf{id}\, r(A)\end{Bmatrix}$, we get

$$m(\mathcal{S}\otimes\mathrm{id})\Delta(A) = \sum_{XY\in\mathcal{H},\, \genfrac{}{}{0pt}{}{A}{Y}\in\mathcal{V}} \quad \sum_{UV=A,\, \genfrac{}{}{0pt}{}{U^{-1}}{V}=X} \daleth(U^{-1}) \quad X,$$

which gives (2.5) in view of (3.13), after observing that $\daleth(U^h)=\daleth(V)$.

We finally show (2.6). Let $A\in\mathcal{B}$. We have

$$\begin{aligned}
m^{(2)}(\mathcal{S}\otimes\mathrm{id}\otimes\mathcal{S})\Delta^{(2)}(A) &= \sum_{XYZ=A} \frac{\daleth(Y)\daleth(Z)\daleth(X^{-1})\daleth(Z^{-1})}{\daleth(X^h)\daleth(Z^h)} \quad X^{-1}.Y.Z^{-1}\\
&= \Big(\sum \frac{\daleth(Y)\daleth(Z)\daleth(X^{-1})}{\daleth(X^h)}\Big)\frac{\daleth(A^{-1})}{\daleth(A^h)} \quad A^{-1},
\end{aligned}$$

where the last sum runs over all triples (X,Y,Z) satisfying (1.4), (1.5); see Lemma 1.3. Here we have used that $\daleth(A^h)=\daleth(Z^h)$, which follows from (1.4), and $\daleth(A^{-1})=\daleth(Z^{-1})$, which follows from (1.5). Now $\daleth(Y)=$

$\vartheta(bl(Y)) = \vartheta(br(X)) = \vartheta(bl(X^h)) = \daleth(X^h)$ and thus, because of Lemma 1.20 and hypothesis (3.22), we get

$$\sum \frac{\daleth(Y)\daleth(Z)\daleth(X^{-1})}{\daleth(X^h)} = \Big(\sum_{Z\in\mathcal{B}: t(Z)=t(A),\, r(Z)=r(A)} \daleth(Z)\Big) \times \Big(\sum_{X\in\mathcal{B}: b(X)=b(A),\, l(X)=l(A)} \daleth(X^{-1})\Big) = 1,$$

proving (2.6).

Finally, the maps $\Bbbk\mathbf{D} \to (\Bbbk^{(\vartheta)}\mathcal{T})_s$, $D \mapsto {}_D\mathbf{1}$, and $(\Bbbk\mathbf{E})^{\mathrm{op}} \to (\Bbbk^{(\vartheta)}\mathcal{T})_t$, $E \mapsto \mathbf{1}_E$, determine algebra isomorphisms, by Lemma 2.2. □

Remark 3.7. Let $\vartheta : \mathcal{P} \to \Bbbk^\times$ be any function and define $\daleth$ as before. For any $(g, x) \in \mathcal{V}\,{}_t{\times}_r\,\mathcal{H}$, set

$$c(g, x) := \sum_{V\in\mathcal{B}: t(V)=x,\, r(V)=g} \daleth(V).$$

Let us say that $P \in \mathcal{P}$ is *affiliated* to (g, x) if there exists $V \in \mathcal{B}$: $t(V) = x$, $r(V) = g$, $bl(V) = P$. Assume that the filling condition (2.10) holds and that

$$c(g, x) = c(\mathrm{id}_{\mathcal{V}}\, P, \mathrm{id}_{\mathcal{H}}\, P) \neq 0 \tag{3.26}$$

for all (g, x) such that P is affiliated to (g, x). Let $\widetilde{\vartheta} : \mathcal{P} \to \Bbbk^\times$ be given by

$$\widetilde{\vartheta}(P) = \frac{1}{c(\mathrm{id}_{\mathcal{V}}\, P, \mathrm{id}_{\mathcal{H}}\, P)}\vartheta(P), \qquad p \in \mathcal{P}.$$

Then the $\widetilde{\daleth}$ corresponding to $\widetilde{\vartheta}$ satisfies (3.22).

Proof of Theorem 2.3. Consider $\vartheta : \mathcal{P} \to \Bbbk^\times$, $\vartheta(P) = \dfrac{1}{\theta(P)}$. We only need to show that ϑ satisfies (3.22). Let $(g, x) \in \mathcal{V}\,{}_t{\times}_r\,\mathcal{H}$. Because of the filling condition (2.10), there exists $A \in \mathcal{B}$ such that $t(A) = x$, $r(A) = g$; Set $Y = A$, $X = \mathbf{id}\, l(A)$,$X = \mathbf{id}\, b(A)$. Then, by Proposition 1.14, we have

$$\sum_{V\in\mathcal{B}: t(V)=x,\, r(V)=g} \daleth(V) = \sum_{UV=A,\, RS=B,\, \frac{U}{R}=X,\, \frac{V}{S}=Y} \frac{1}{\neg(V)}$$

$$= \sum_{UV=A,\, RS=B,\, \frac{U}{R}=X,\, \frac{V}{S}=Y} \frac{1}{\neg(t(Y), r(A))} = 1.$$

□

Let τ be given by a general ϑ satisfying (3.22). The following statement generalizes Proposition 2.7.

Proposition 3.8. *$\Bbbk^{(\vartheta)}\mathcal{T}$ is a regular weak Hopf algebra if and only if ϑ is constant along the connected components of $\mathcal{P}$ defined by $\mathbf{D}$.*

Proof. Let $A \in \mathcal{B}$. Then

$$\mathcal{S}^2(A) = \frac{\daleth(A)\daleth(A^{-1})}{\daleth(A^h)\daleth(A^v)}A = \frac{\vartheta(bl(A))\vartheta(tr(A))}{\vartheta(br(A))\vartheta(tl(A))}A. \tag{3.27}$$

It follows that $\mathcal{S}^2({}_D\mathbf{1}) = \frac{\vartheta(e(D))}{\vartheta(s(D))}{}_D\mathbf{1}$, $D \in \mathbf{D}$. This proves the proposition. $\square$

Remark 3.9. Let $D \in \mathbf{D}$ and $E \in \mathbf{E}$. Then, as in Lemma 2.7,

$$\mathcal{S}(\mathbf{1}_E) = {}_{E^{-1}}\mathbf{1}, \qquad \mathcal{S}({}_D\mathbf{1}) = \frac{\vartheta(e(D))}{\vartheta(s(D))}\,\mathbf{1}_{D^{-1}}. \tag{3.28}$$

Proposition 3.10. [2] *There is a pivotal group-like element $G \in \Bbbk^{(\vartheta)}\mathcal{T}$ implementing $\mathcal{S}^2$ by conjugation– see* [BNSz, N2]. *Explicitly,*

$$G = \sum_{x\in\mathcal{H}} \frac{\vartheta(l(x))}{\vartheta(r(x))}\,\mathrm{id}\,x,$$

where $\mathcal{H}(P,Q)$ is the set of all horizontal arrows going from P to Q.

Notice however that neither the elements $\mathbf{1}_{\Theta(P)}$, $P \in \mathcal{P}$, nor the boxes $\mathrm{id}\,x$, $x \in \mathcal{H}$, are central in $\Bbbk^{(\vartheta)}\mathcal{T}$.

Proof. Note that $G = \mathcal{S}(w)w^{-1}$, where $w = \sum_{P\in\mathcal{P}} \dfrac{1}{\vartheta(P)}\,\mathbf{1}_{\Theta(P)}$. Indeed,

$$w^{-1} = \sum_{P\in\mathcal{P}} \vartheta(P)\,\mathbf{1}_{\Theta(P)} \qquad \text{and} \qquad \mathcal{S}(w) = \sum_{Q\in\mathcal{P}} \frac{1}{\vartheta(Q)}\,{}_{\Theta(Q)}\mathbf{1},$$

the first equality by Lemma 2.2 and the second by (3.28). Hence

$$\begin{aligned} G &= \sum_{P,Q\in\mathcal{P}} \frac{\vartheta(P)}{\vartheta(Q)}\,\mathbf{1}_{\Theta(P)}\cdot{}_{\Theta(Q)}\mathbf{1} \\ &= \sum_{P,Q\in\mathcal{P}} \frac{\vartheta(P)}{\vartheta(Q)} \sum_{x\in\mathcal{H}(P,Q)} \mathrm{id}\,x \\ &= \sum_{x\in\mathcal{H}} \frac{\vartheta(l(x))}{\vartheta(r(x))}\,\mathrm{id}\,x. \end{aligned}$$

[2]We thank the referee for suggesting the formula for the pivotal element included here.

A straightforward computation shows that $\mathcal{S}^2(A) = G^{-1}.A.G$, $A \in \mathcal{B}$. Finally,

$$\begin{aligned}\Delta(G) &= \Delta(1) \sum_{x\in\mathcal{H}} \sum_{z,w\in\mathcal{H}: zw=x} \frac{\vartheta(l(x))}{\vartheta(r(x))} \operatorname{id} z \otimes \operatorname{id} w \\ &= \Delta(1) \sum_{z,w\in\mathcal{H}} \frac{\vartheta(l(z))}{\vartheta(r(z))} \frac{\vartheta(l(w))}{\vartheta(r(w))} \operatorname{id} z \otimes \operatorname{id} w \\ &= \Delta(1)(G \otimes G).\end{aligned}$$

Here the first equality is by (2.24), and the second uses that $l(z) = l(x)$, $r(w) = r(x)$ and $r(z) = l(w)$ whenever $x = zw$. ☐

3.3. C^*-structure.

In this subsection $\Bbbk = \mathbb{C}$. Let $\vartheta : \mathcal{P} \to \mathbb{R}^+$ be a function satisfying condition (3.22), and let $\mathbb{C}^{(\vartheta)}\mathcal{T}$ be the corresponding weak Hopf algebra. We describe a C^*-structure on $\mathbb{C}^{(\vartheta)}\mathcal{T}$. See [BNSz].

Consider the map $\lambda : \mathcal{B} \to \mathbb{R}^+$, given by

$$\lambda(A) = \frac{\daleth(A^h)}{\daleth(A^{-1})} = \frac{\vartheta(br(A))}{\vartheta(rt(A))}.$$

Lemma 3.11. *(i) λ defines a character on the vertical composition groupoid $\mathcal{B} \rightrightarrows \mathcal{H}$.*

(ii) λ is invariant with respect to horizontal composition, that is, $\lambda(UV) = \lambda(V)$, for all $U, V \in \mathcal{B}$ such that $U|V$.

Proof. (i) Let $A, B \in \mathcal{B}$ such that $\frac{A}{B}$. We have $br(A) = rt(B)$. Therefore,

$$\lambda(A)\lambda(B) = \frac{\daleth(A^h)}{\daleth(A^{-1})} \frac{\daleth(B^h)}{\daleth(B^{-1})} = \frac{\vartheta(br(A))}{\vartheta(rt(A))} \frac{\vartheta(br(B))}{\vartheta(rt(B))} = \frac{\vartheta(br(B))}{\vartheta(rt(A))}.$$

Since $br\left(\frac{A}{B}\right) = br(B)$ and $rt\left(\frac{A}{B}\right) = rt(A)$, the equality $\lambda\left(\frac{A}{B}\right) = \lambda(A)\lambda(B)$ follows. This proves part (i).

(ii) Let $U, V \in \mathcal{B}$ such that $U|V$. We have $br(A) = rt(B)$. Hence, $\lambda(UV) = \frac{\vartheta(br(UV))}{\vartheta(rt(UV))} = \frac{\vartheta(br(V))}{\vartheta(rt(V))} = \lambda(V)$, proving part (ii). ☐

Proposition 3.12. *$\mathbb{C}^{(\vartheta)}\mathcal{T}$ is a C^*-quantum groupoid with respect to the involution $A^* := \lambda(A)A^v$.*

Proof. We have $\lambda(A^v) = \lambda(A)^{-1}$. This implies that $*$ is an involution, since λ takes real values. Condition $(A.B)^* = B^*.A^*$ follows from Lemma 3.11(i)

and the fact that $\begin{pmatrix} A \\ B \end{pmatrix}^v = \begin{pmatrix} B^v \\ A^v \end{pmatrix}$ for composable boxes A, B. Also, for all $A \in \mathcal{B}$,

$$\begin{aligned}\Delta(A^*) &= \lambda(A) \sum_{XY=A^v} \frac{1}{\urcorner(Y)} X \otimes Y = \lambda(A) \sum_{UV=A} \frac{1}{\urcorner(V^v)} U^v \otimes V^v \\ &= \sum_{UV=A} \frac{\lambda(V)}{\urcorner(V^v)} U^v \otimes V^v = \sum_{UV=A} \frac{1}{\lambda(U)\urcorner(V^v)} U^* \otimes V^*,\end{aligned}$$

by Lemma 3.11(ii). Since $rt(U) = lt(V)$ and $br(U) = bl(V)$, we have

$$\lambda(U)\urcorner(V^v) = \frac{\vartheta(br(U))\,\vartheta(tl(V))}{\vartheta(rt(U))} = \vartheta(br(U)) = \urcorner(V).$$

Thus $\Delta(A^*) = \Delta(A)^{*\otimes *}$.

Let $\varphi : \mathcal{B} \to \mathbb{R}^+$ be given by $\varphi(A) = \begin{cases} 1, & \text{if } A \in \mathcal{H}, \\ 0, & \text{otherwise}, \end{cases}$ $A \in \mathcal{B}$. Let also $(\ \ |\ \ ) : \mathbb{C}^{(\vartheta)}\mathcal{T} \times \mathbb{C}^{(\vartheta)}\mathcal{T} \to \mathbb{C}$ be the unique inner product such that $(A|B) = \varphi(A^*.B)$, $A, B \in \mathcal{B}$. This makes $\mathbb{C}^{(\vartheta)}\mathcal{T}$ into a C^*-quantum groupoid. $\square$

3.4. **Duality.**

Let $\vartheta : \mathcal{P} \to \Bbbk^\times$ be a generalized corner function satisfying (3.22), and let $\Bbbk^{(\vartheta)}\mathcal{T}$ be the corresponding weak Hopf algebra. We describe in this subsection the dual weak Hopf algebra $(\Bbbk^{(\vartheta)}\mathcal{T})^*$.

Consider the map $\mu : \mathcal{B} \to \Bbbk^\times$, defined by

$$\mu(A) = \frac{\urcorner(A^v)}{\urcorner(A^{-1})} = \frac{\vartheta(lt(A))}{\vartheta(rt(A))}.$$

The following properties are analogous to those in Lemma 3.11.

Lemma 3.13. *(i) μ defines a character on the horizontal composition groupoid $\mathcal{B} \rightrightarrows \mathcal{V}$.*

(ii) μ is invariant with respect to vertical composition. $\square$

Recall that the transpose double groupoid $\mathcal{T}^t = \begin{matrix} \mathcal{B} & \rightrightarrows & \mathcal{V} \\ \downdownarrows & & \downdownarrows \\ \mathcal{H} & \rightrightarrows & \mathcal{P} \end{matrix}$ is obtained from $\mathcal{T}$ interchanging 'horizontal' and 'vertical' throughout. For $A \in \mathcal{B}$, let A^t denote the same box in the transpose double groupoid; so that

$$A^t = t(A) \underset{r(A)}{\overset{l(A)}{\square}} b(A).$$

Proposition 3.14. $(\Bbbk^{(\vartheta)}\mathcal{T})^* \simeq \Bbbk^{(\vartheta)}\mathcal{T}^t$ *as weak Hopf algebras.*

Proof. Recall the basis $\{\underline{A}\}_{A\in\mathcal{B}}$ of $\Bbbk\mathcal{T}$ introduced in Remark 2.5. Then the map $(\ ,\ ):\Bbbk^{(\vartheta)}\mathcal{T}\otimes\Bbbk^{(\vartheta)}\mathcal{T}^t\to\Bbbk$ defined by $(\underline{A},B):=\mu(A)\delta_{A,B^t}$ is a non-degenerate weak Hopf algebra pairing. □

3.5. **Fusion categories.**

We next establish necessary and sufficient conditions for $\operatorname{Rep}\Bbbk^{(\vartheta)}\mathcal{T}$ to be a fusion category. For this, we study the $\Bbbk^{(\vartheta)}\mathcal{T}$-module structure of the target subalgebra $\Bbbk^{(\vartheta)}\mathcal{T}_t$. We begin by the following general result.

Let $\mathcal{G}$ be a groupoid with base $\mathcal{P}$ and let $\sim$ be the corresponding equivalence relation on $\mathcal{P}$: for $P,Q\in\mathcal{P}$, $P\sim Q$ iff there exists $g\in\mathcal{G}$ such that $s(g)=P$, $e(g)=Q$.

Assume that $\mathcal{G}$ acts on a fiber bundle $\gamma:\mathcal{E}\to\mathcal{P}$ via $\rightharpoondown:\mathcal{G}\,{}_e\!\times_\gamma\mathcal{E}\to\mathcal{E}$. If $P\in\mathcal{P}$, then we denote by $\mathcal{E}_P$ the fiber $\gamma^{-1}(P)$. Clearly, $\gamma(\mathcal{E})$ is stable under the equivalence relation $\sim$; for, the action $g\rightharpoondown\ :\mathcal{E}_{e(g)}\to\mathcal{E}_{s(g)}$ is a bijection. We say that the action is *transitive* if $\gamma(\mathcal{E})$ is just one class for the relation $\sim$.

Let $\lambda:\mathcal{G}\to\Bbbk^\times$ be a character, *i. e.* $\lambda(gh)=\lambda(g)\lambda(h)$ when $e(g)=s(h)$. Let $\Bbbk_\lambda\mathcal{E}$ be the $\Bbbk\mathcal{G}$-module with $\Bbbk$-basis (x_E), $E\in\mathcal{E}$, and action

$$g.x_E=\begin{cases}\lambda(g)x_{g\rightharpoondown E} & \text{if } e(g)=\gamma(E),\\ 0 & \text{if not.}\end{cases}$$

Lemma 3.15. *The following are equivalent.*

(i) The $\Bbbk\mathcal{G}$-module $\Bbbk_\lambda\mathcal{E}$ is simple.

(ii) The action is transitive and $\#\mathcal{E}_P=1$ for any $P\in\gamma(\mathcal{E})$.

Proof. (i) $\implies$ (ii). If $P\in\gamma(\mathcal{E})$, we set

$$x_P=\sum_{E\in\mathcal{E}_P}x_E.$$

If $g\in\mathcal{G}$ has $s(g)=P$, $e(g)=Q$, then $g.x_Q=\lambda(g)x_P$. Let $\mathcal{Q}\subset\mathcal{P}$ be an equivalence class of $\sim$ and let M be the $\Bbbk$-span of the elements x_P, $P\in\mathcal{Q}$. Then

$$\oplus_{P\in\mathcal{Q}}\Bbbk x_P=M=\Bbbk_\lambda\mathcal{E}=\oplus_{P\in\mathcal{P}}\Bbbk\mathcal{E}_P,$$

the second equality since $\Bbbk_\lambda\mathcal{E}$ is simple. This implies that $\mathcal{Q}=\mathcal{P}$, *i. e.* that the action is transitive; and that $\dim\Bbbk\mathcal{E}_P=1$, *i. e.* that $\#\mathcal{E}_P=1$, for any $P\in\gamma(Q)$.

(ii) $\implies$ (i). Let M be a non-zero $\Bbbk\mathcal{G}$-submodule of $\Bbbk_\lambda\mathcal{E}$ and let $0 \neq m = \sum_{E\in\mathcal{P}} m_E x_E \in M$, where $m_E \in \Bbbk$. Fix E such that $m_E \neq 0$; then $x_E = m_E^{-1} \operatorname{id}\gamma(E).m \in M$, hence $\Bbbk\mathcal{E}_{\gamma(E)} \subset M$ by the assumption "$\#\mathcal{E}_P = 1$ for any $P \in \gamma(\mathcal{E})$". Now the transitivity assumption implies that $\Bbbk\mathcal{E} \subset M$. Hence $\Bbbk_\lambda\mathcal{E}$ is simple. $\square$

Recall that a semisimple finite tensor category is called a fusion category exactly when the unit object is simple [ENO]. A semisimple weak Hopf algebra is called *connected* if its representation category is fusion [ENO, Section 4].

Proposition 3.16. *The tensor category* $\operatorname{Rep}\Bbbk^{(\vartheta)}\mathcal{T}$ *is a fusion category (or* $\Bbbk^{(\vartheta)}\mathcal{T}$ *is connected) if and only if the following hold:*

(a) $\mathcal{V} \rightrightarrows \mathcal{P}$ *is connected,* cf. *Remark 1.9.*
(b) *For any* $x \in \mathcal{H}$*, there exists at most one* $E \in \mathbf{E}$ *such that* $b(E) = x$*.*

Proof. Let $A \in \mathcal{B}$ and $E \in \mathbf{E}$. Then the action of A on $\Bbbk\mathcal{T}_t$ is given in terms of the action $\curvearrowright$ in (1.14). Explicitly,

$$\begin{aligned} A \cdot \mathbf{1}_E &= \epsilon_t(A.\mathbf{1}_E) = \epsilon_t(E \rightharpoondown A) \\ &= \begin{cases} \dfrac{\urcorner((E \rightharpoondown A)^{-1})}{\urcorner((E \rightharpoondown A)^h)} \mathbf{1}_{\psi(E\rightharpoondown A)} = \dfrac{\urcorner(A^{-1})}{\urcorner(A^h)} \mathbf{1}_{A\curvearrowright E}, & \text{if } b(E \rightharpoondown A) \in \mathcal{P}, \\ 0 & \text{if not.} \end{cases} \end{aligned}$$

Note that $b(E \rightharpoondown A) = b(E)b(A)$, thus $b(E \rightharpoondown A) \in \mathcal{P}$ iff $b(E) = b(A)^{-1}$.

We can then apply Proposition 3.15 with $\gamma(E) = b(E)^{-1}$, $E \in \mathbf{E}$. The action $\curvearrowright$ is transitive iff (a) holds, by Lemma 1.10; and condition (b) is equivalent to "$\#\mathbf{E}_x = 1$ for any $x \in \gamma(\mathbf{E})$". $\square$

In particular, if $\operatorname{Rep}\Bbbk^{(\vartheta)}\mathcal{T}$ is fusion then necessarily $\mathcal{P}$ is vertically connected, because of condition (a) applied to the boxes $\Theta(P) \in \mathbf{E}$.

3.6. **Frobenius-Perron dimensions of simple objects.**

Let $\mathcal{T}$ be a finite double groupoid and ϑ a generalized corner function satisfying (3.22). Let $\operatorname{Rep}\Bbbk^{(\vartheta)}\mathcal{T}$ be the corresponding weak Hopf algebra.

Let $\mathcal{B} \rightrightarrows \mathcal{H}$ be the vertical composition groupoid. Let $\mathcal{R}$ be the equivalence relation on $\mathcal{H}$ defined by $\mathcal{B}$ and let X^2 be the coarse groupoid on $X \in \mathcal{R}$. We fix $x \in X$, for $X \in \mathcal{R}$, and denote by $\mathcal{B}(x) = \mathcal{B}(x, x)$ the group

of loops in x. If $y \in \mathcal{H}$ belongs to the class X, then we denote $\overline{y} = x$. The groupoid structure of $\mathcal{B}$ is determined by

$$\mathcal{B} \simeq \coprod_{X \in \mathcal{R}} X^2 \times \mathcal{B}(x). \tag{3.29}$$

If F is a finite set, we denote by $\Bbbk F$ the vector space with base F and by $M_F(\Bbbk)$ the 'matrix algebra' $\operatorname{End}(\Bbbk F)$, with matrix idempotents E_{st}, $s, t \in F$. Then (3.29) gives in turn an isomorphism of algebras

$$\Bbbk\mathcal{T} \simeq \prod_{X \in \mathcal{R}} M_X(\Bbbk) \otimes \Bbbk\mathcal{B}(x)$$

that maps the identity arrows to primitive idempotents:

$$\operatorname{id} y \mapsto E_{yy} \otimes e \in M_X(\Bbbk) \otimes \Bbbk\mathcal{B}(x),$$

$y \in X$, $X \in \mathcal{R}$. Clearly any simple $\Bbbk^{(\vartheta)}\mathcal{T}$-module is of the form $U = \Bbbk X \otimes V$, where $X \in \mathcal{R}$ and V is a simple $\mathcal{B}(x)$-module.

Suppose for the rest of this subsection that $\operatorname{Rep} \Bbbk^{(\vartheta)}\mathcal{T}$ *is a fusion category (or that* $\Bbbk^{(\vartheta)}\mathcal{T}$ *is connected), see Proposition 3.16; and that* $\Bbbk = \mathbb{C}$. Recall the pivotal group-like element G computed in Proposition 3.10. Then $\operatorname{Rep} \Bbbk^{(\vartheta)}\mathcal{T}$ is a pivotal fusion category [ENO, Definition 2.7] and the quantum dimension (corresponding to this pivotal structure) of the simple module $U = \Bbbk X \otimes V$ is

$$\mathbf{dim}\ U = \frac{\operatorname{tr}_U(G)}{\dim \Bbbk^{(\vartheta)}\mathcal{T}_t},$$

thus

$$\mathbf{dim}\ U = \frac{1}{\#\mathbf{E}} \sum_{y \in X} \frac{\vartheta(l(y))}{\vartheta(r(y))} \operatorname{tr}_U(\operatorname{id} y) = \frac{\dim V}{\#\mathbf{E}} \sum_{y \in X} \frac{\vartheta(l(y))}{\vartheta(r(y))}. \tag{3.30}$$

Therefore, the *quantum dimension* $\mathbf{dim}\ \Bbbk^{(\vartheta)}\mathcal{T}$ is given by

$$\begin{aligned}\mathbf{dim}\ \Bbbk^{(\vartheta)}\mathcal{T} &= \sum_{X \in \mathcal{R}, V \in \widehat{\mathcal{B}(x)}} \Big(\frac{\dim V}{\#\mathbf{E}}\Big)^2 \Big| \sum_{y \in X} \frac{\vartheta(l(y))}{\vartheta(r(y))} \Big|^2 \\ &= \frac{1}{(\#\mathbf{E})^2} \sum_{X \in \mathcal{R}} \Big| \sum_{y \in X} \frac{\vartheta(l(y))}{\vartheta(r(y))} \Big|^2 |\mathcal{B}(x)|,\end{aligned}$$

cf. [ENO, Definition 2.2 and Proposition 2.9].

Lemma 3.17. *If* $\sum_{y \in X} \dfrac{\vartheta(l(y))}{\vartheta(r(y))} > 0$ *for any class* $X \in \mathcal{R}$, *then the Frobenius-Perron dimensions of the simple* $\Bbbk^{(\vartheta)}\mathcal{T}$*-modules agree with their quantum dimensions, hence they are given by* (3.30).

Proof. This follows because FP-dim is the unique ring homomorphism from the Grothendieck ring to the complex numbers taking positive real values on the irreducible modules. □

When ϑ takes values in $\mathbb{Q}_{>0}$ (for example, when $\vartheta = \dfrac{1}{\theta}$ comes from the corner function), Lemma 3.17 says that the Frobenius-Perron dimension of a simple $\Bbbk^{(\vartheta)}\mathcal{T}$-module U is a rational number; since by [ENO] FP-dim U is an algebraic integer, it is an integer. That is, the Frobenius-Perron dimension of the simple $\Bbbk\mathcal{T}$-modules are integers:

$$\frac{\dim V}{\#\mathbf{E}} \sum_{y \in X} \frac{\theta(r(y))}{\theta(l(y))} \in \mathbb{N},$$

for any class $X \in \mathcal{R}$, for any irreducible representation V of $\mathcal{B}(x)$. In particular $\operatorname{Rep} \Bbbk\mathcal{T}$ is tensor equivalent to the representation category of a finite-dimensional semisimple quasi-Hopf algebra [ENO, Theorem 8.33].

In general, a connected weak Hopf algebra H is called *pseudo-unitary* if $\mathbf{dim}\ H$ coincides with the Frobenius-Perron dimension of H.

Proposition 3.18. *Assume that*

$$\frac{\urcorner(A)\urcorner(A^{-1})}{\urcorner(A^h)\urcorner(A^v)} = \frac{\vartheta(lb(A))\vartheta(rt(A))}{\vartheta(rb(A))\vartheta(lt(A))} > 0 \tag{3.31}$$

for any box $A \in \mathcal{B}$. Then $\Bbbk^{(\vartheta)}\mathcal{T}$ is a pseudo-unitary weak Hopf algebra and for any irreducible $\Bbbk^{(\vartheta)}\mathcal{T}$-module $U = \Bbbk X \otimes V$ as above,

$$\text{FP-dim } U = \frac{\dim V}{\#\mathbf{E}} \Big| \sum_{y \in X} \frac{\vartheta(l(y))}{\vartheta(r(y))} \Big|.$$

Proof. By (3.27), condition (3.31) says that all eigenvalues of $\mathcal{S}^2$ are strictly positive numbers; the first claim follows from the criterion given in [N2, Corollary 5.2.5]. The second claim follows from [ENO, Proposition 8.21]. □

4. Examples

4.1. **Matched pairs.** [AN].

Given a groupoid $\mathcal{G}$ one can consider the double groupoid whose boxes are commuting squares in $\mathcal{G}$. More generally, if $\mathcal{H}$ and $\mathcal{V}$ are wide subgroupoids of $\mathcal{G}$, there is a double groupoid of commuting squares in $\mathcal{G}$ whose horizontal arrows belong to $\mathcal{H}$ and whose vertical arrows belong to $\mathcal{V}$. A special case of this remark is given by the following construction.

Let $\triangleleft : \mathcal{H}\,{}_r\!\times_t \mathcal{V} \to \mathcal{H}$, $\triangleright : \mathcal{H}\,{}_r\!\times_t \mathcal{V} \to \mathcal{V}$ be a matched pair of finite groupoids, on the same basis $\mathcal{P}$. Here, l, t (respectively, r, b) denote the source and target maps of $\mathcal{H}$ and $\mathcal{V}$. This is equivalent to giving an exact factorization $\mathcal{G} = \mathcal{V}_b \times_l \mathcal{H}$.

Let $\mathcal{B} := \mathcal{V}\,{}_t{\times}_r\,\mathcal{H}$. We have a double groupoid

$$\begin{matrix} \mathcal{V}\,{}_t{\times}_r\,\mathcal{H} & \rightrightarrows & \mathcal{H} \\ \downdownarrows & & \downdownarrows \\ \mathcal{V} & \rightrightarrows & \mathcal{P}, \end{matrix}$$

defined as follows: $\mathcal{V}\,{}_t{\times}_r\,\mathcal{H} \rightrightarrows \mathcal{H}$ and $\mathcal{V}\,{}_t{\times}_r\,\mathcal{H} \rightrightarrows \mathcal{V}$ are, respectively, the transformation groupoids corresponding to the actions $\triangleleft$ and $\triangleright$. These examples exhaust the class of vacant double groupoids, as shown by Mackenzie [Ma1]. The corresponding weak Hopf algebras have been introduced in [AN].

The core groupoid of a vacant double groupoid is the discrete groupoid on $\mathcal{P}$. Therefore the source and target subalgebras of the associated quantum groupoid are commutative, *i.e.*, this quantum groupoid is a *face algebra* in the sense of Hayashi [H].

For Hopf algebras coming from matched pairs of finite groups, the pictorial description using boxes has been used in [Mj, T].

4.2. **Bimodules over a separable algebra.**

Let $\mathcal{G}$ be a finite groupoid with source and target maps $s, e : \mathcal{G} \rightrightarrows \mathcal{P}$. There is a double groupoid

$$\mathcal{T} = \mathcal{T}(\mathcal{G}) = \begin{matrix} \mathcal{G}\times\mathcal{G} & \rightrightarrows & \mathcal{P}\times\mathcal{P} \\ \downdownarrows & & \downdownarrows \\ \mathcal{G} & \rightrightarrows & \mathcal{P} \end{matrix},$$

canonically associated to $\mathcal{G}$, where

- $\mathcal{P}\times\mathcal{P} \rightrightarrows \mathcal{P}$ is the coarse groupoid on $\mathcal{P}$,
- $\mathcal{G}\times\mathcal{G} \rightrightarrows \mathcal{G}$ is the coarse groupoid on $\mathcal{G}$,
- $\mathcal{G}\times\mathcal{G} \rightrightarrows \mathcal{P}\times\mathcal{P}$ is the product groupoid of $\mathcal{G}$ with itself.

See [BMa, Example 1.3]. A box in this double groupoid is a pair (g, h) of arrows in $\mathcal{G}$ and it can be depicted as

$$(g,h) = g\,\square\,h = g \begin{array}{c} s(g) \quad s(h) \\ \boxed{(g,h)} \\ e(g) \quad e(h) \end{array} h.$$

Horizontal and vertical compositions are defined, respectively, by

$$g\,\square\,h \quad h\,\square\,t = g\,\square\,t, \qquad \begin{matrix} g\,\square\,h \\ g'\,\square\,h' \end{matrix} = gg'\,\square\,hh',$$

for all pairs of composable arrows g, g' and h, h', and for all arrow t. The horizontal and vertical identities are given by

$$\mathbf{id} g = g \,\square\, g, \qquad \mathbf{id}(p,q) = \mathrm{id}_p \,\square\, \mathrm{id}_q,$$

for all $g \in \mathcal{G}$, $p, q \in \mathcal{P}$.

We summarize the relevant facts on the structure of this double groupoid.

Lemma 4.1. *(a). The core groupoid* $\mathbf{D}$ *is isomorphic to* $\mathcal{G}$.

(b). Let $g, h \in \mathcal{G}$ *and let* $A = g \,\square\, h \in \mathcal{B}$. *Then*

(i) $\urcorner(A) = \#\{u \in \mathcal{G} : \ s(u) = s(g)\}$.
(ii) $\llcorner(A) = \#\{u \in \mathcal{G} : \ e(u) = e(h)\}$.
(iii) $\ulcorner(A) = \#\{u \in \mathcal{G} : \ s(u) = s(h)\}$.
(iv) $\lrcorner(A) = \#\{u \in \mathcal{G} : \ e(u) = e(g)\}$. □

By Lemma 4.1, the source subalgebra of the corresponding weak Hopf algebra $\Bbbk\mathcal{T}(\mathcal{G})$ is isomorphic to $\Bbbk\mathcal{G}$. Also, since $\urcorner(A) = \lrcorner(A)$, and $\llcorner(A) = \ulcorner(A)$, for all $A \in \mathcal{B}$, $\mathcal{S}^2 = \mathrm{id}$ in $\Bbbk\mathcal{T}(\mathcal{G})$, *cf.* Lemma 2.6.

Remark 4.2. Observe that in the double groupoid $\mathcal{T}(\mathcal{G}) \coprod {}^t\mathcal{T}(\mathcal{G})$, where ${}^t\mathcal{T}(\mathcal{G})$ is the transpose double groupoid, the corner functions give four pair-wise *distinct* maps $\mathcal{B} \to \mathbb{N}$.

Proposition 4.3. *The category* $\operatorname{Rep} \Bbbk\mathcal{T}(\mathcal{G})$ *is equivalent to the category* ${}_{\Bbbk\mathcal{G}}\mathcal{M}_{\Bbbk\mathcal{G}}$ *of finite dimensional* $\Bbbk\mathcal{G}$*-bimodules.*

Proof. The algebra $\Bbbk\mathcal{G}$ is a separable algebra, with separability idempotent

$$e = \sum_{g \in \mathcal{G}} \frac{1}{d(s(g))}\, g \otimes g^{-1} \ \in \Bbbk\mathcal{G} \otimes (\Bbbk\mathcal{G})^{\mathrm{op}},$$

where $d(s(g))$ is the number of arrows of $\mathcal{G}$ having source $s(g)$. Hence, for all $\Bbbk\mathcal{G}$-bimodules V and W, the inclusion $e.(V \otimes_{\Bbbk} W) \subseteq V \otimes_{\Bbbk} W$ induces a natural isomorphism

$$V \otimes_{\Bbbk\mathcal{G}} W \simeq e.(V \otimes_{\Bbbk} W).$$

We show in what follows that this coincides with the tensor product of $\Bbbk\mathcal{T}(\mathcal{G})$-modules. As an algebra, $\Bbbk\mathcal{T}(\mathcal{G})$ is isomorphic to $\Bbbk(\mathcal{G} \times \mathcal{G})$; this is in turn isomorphic to $\Bbbk\mathcal{G} \otimes (\Bbbk\mathcal{G})^{\mathrm{op}}$, the isomorphism given by inverting the second factor. This gives a natural equivalence between $\Bbbk\mathcal{T}(\mathcal{G})$-modules and $\Bbbk\mathcal{G}$-bimodules. Tensor product of two $\Bbbk\mathcal{T}(\mathcal{G})$-modules V and W is defined by $V \otimes W = \Delta(1).(V \otimes_{\Bbbk} W)$. Explicitly, we have

$$\begin{aligned}\Delta(1) &= \sum_{p,q\in\mathcal{P},h\in\mathcal{G}} \frac{1}{d(s(h))} \quad \mathrm{id}_p \boxed{}\, h \otimes h \boxed{}\, \mathrm{id}_q \\ &= \sum_{h\in\mathcal{G}} \frac{1}{d(s(h))} \quad \left(\sum_{p\in\mathcal{P}} \mathrm{id}_p \boxed{}\, h\right) \otimes \left(\sum_{q\in\mathcal{P}} h \boxed{}\, \mathrm{id}_q\right),\end{aligned}$$

which corresponds to the element $e \in \Bbbk\mathcal{G} \otimes (\Bbbk\mathcal{G})^{\mathrm{op}}$. Therefore $V \otimes W$ corresponds to $V \otimes_{\Bbbk\mathcal{G}} W$ as linear spaces. Finally, the action of a box $g \boxed{}\, h$ on this tensor product coincides with the action of the corresponding element in $\Bbbk\mathcal{G} \otimes (\Bbbk\mathcal{G})^{\mathrm{op}}$. This proves the lemma. □

Observe that any algebra R which is separable over $\Bbbk$ is isomorphic as an algebra, to the groupoid algebra of a (not canonical) finite groupoid. The following proposition is a consequence of Lemma 4.3.

Proposition 4.4. *Let R be separable algebra over $\Bbbk$. There exists a finite groupoid $\mathcal{G}$ such that ${}_R\mathcal{M}_R$ is tensor equivalent to* $\operatorname{Rep} \Bbbk\mathcal{T}(\mathcal{G})$. □

4.3. **Group theoretical fusion categories.**

In this section we consider the group theoretical categories introduced by Ostrik in [O]. Let $\mathrm{Vec} := \mathrm{Vec}_{\Bbbk}$ denote the category of vector spaces over $\Bbbk$.

Let G be a finite group, and let $F \subseteq G$ be a subgroup. There is a fusion category $\mathcal{C}(G,\omega,F,\alpha)$, called a *group theoretical* category [ENO, Definition 8.46], associated to the following data:

- a normalized 3-cocycle $\omega : G \times G \times G \to k^{\times}$;
- a normalized 2-cochain $\alpha : F \times F \to k^{\times}$;

subject to the condition

$$\omega|_{F\times F\times F} = d\alpha. \tag{4.1}$$

By (4.1), the twisted group algebra $k_\alpha F$ is an (associative unital) algebra in Vec_ω^G of G-graded vector spaces with associativity defined by ω. The category $\mathcal{C}(G,\omega,F,\alpha)$ is by definition the k-linear monoidal category ${}_{\Bbbk_\alpha F}(\mathrm{Vec}_\omega^G)_{\Bbbk_\alpha F}$ of $k_\alpha F$-bimodules in Vec_ω^G: tensor product is $\otimes_{k_\alpha F}$ and the unit object is $k_\alpha F$. A (quasi)-Hopf algebra A is called group theoretical if the category $\operatorname{Rep} A$ of its finite dimensional representations is group theoretical. Every group theoretical category is the representation category of a semisimple finite dimensional quasi-Hopf algebra.

It is shown in [Na] that there is a 3-cocycle $\widetilde{\omega}$, cohomologous to ω, such that the categories $\mathcal{C}(G,\omega,F,\alpha)$ and $\mathcal{C}(G,\widetilde{\omega},F,1)$ are equivalent. That is, up to tensor equivalence, it is enough to consider the case $\alpha = 1$.

4.4. The category Vec_ω^G.

In the next example we shall consider the double groupoid

$$\mathcal{T}_0 = \begin{array}{ccc} G\times G\times G & \rightrightarrows & G\times G \\ \downdownarrows & & \downdownarrows \\ G\times G & \rightrightarrows & G \end{array},$$

where

- the horizontal groupoid $G\times G\times G \rightrightarrows G\times G$ is the direct product of the coarse groupoid on G and the discrete groupoid on G; that is, the double groupoid corresponding to the equivalence relation defined on $G\times G$ by $(x,y)\sim(x',y')$ if and only if $y=y'$;
- the vertical groupoid $G\times G\times G \rightrightarrows G\times G$ is the transformation groupoid associated to the regular action $. : G\times G\times G \to G\times G$, $(x,y).g=(xg,yg)$;
- the horizontal groupoid $G\times G \rightrightarrows G$ is the coarse groupoid on G;
- the vertical groupoid $G\times G \rightrightarrows G$ is the transformation groupoid corresponding to the right regular action of G on itself.

Observe that $\mathcal{T}_0$ is a *vacant* double groupoid, with boxes determined by

$$(a,b,g) := \begin{array}{ccc} a & & b \\ & \square & \\ ag & & bg \end{array},$$

for all $a,b,g\in G$. Horizontal and vertical compositions are as follows:

$$\begin{array}{ccc} a & & b \\ & \square & \\ ag & & bg \end{array} \quad \begin{array}{ccc} b & & c \\ & \square & \\ bg & & cg \end{array} = \begin{array}{ccc} a & & c \\ & \square & \\ ag & & cg \end{array}, \qquad \begin{array}{ccc} a & & b \\ & \square & \\ ag & & bg \\ & \square & \\ agh & & bgh \end{array} = \begin{array}{ccc} a & & b \\ & \square & \\ agh & & bgh \end{array}.$$

Let $\sigma : \mathcal{B}\times_{b,t}\mathcal{B} \to \Bbbk^\times$ be given by

$$\sigma\left(\begin{array}{ccc} a & & b \\ & \square & \\ ag & & bg \end{array}, \begin{array}{ccc} ag & & bg \\ & \square & \\ agh & & cgh \end{array} \right) = \frac{\omega(a,g,h)}{\omega(b,g,h)}. \tag{4.2}$$

The 3-cocycle condition on ω implies that σ is a normalized vertical 2-cocycle for $\mathcal{T}_0$; see [AN]. Moreover, for all boxes A, B, C, D such that $\begin{array}{c|c} A & B \\ \hline C & D \end{array}$, we have $\sigma(AB,CD)=\sigma(A,C)\sigma(B,D)$. Thus σ and the trivial cocycle $\tau=1$ are compatible in the sense of [AN, Definition 3.6], and therefore there is an associated weak Hopf algebra $\Bbbk_\sigma\mathcal{T}_0$.

Proposition 4.5. $\operatorname{Rep}\Bbbk_\sigma\mathcal{T}_0 \simeq \mathrm{Vec}_\omega^G$ *as tensor categories.*

In particular, the Drinfeld center $\mathcal{Z}(\mathrm{Vec}_\omega^G)$ is equivalent to the representation category of the quantum double $D(\Bbbk_\sigma\mathcal{T}_0)$.

Remark 4.6. Compare this example with the weak Hopf algebra $\mathcal{A}^\omega G$ described in [BSz, Appendix].

Proof. As an algebra, $\Bbbk_\sigma \mathcal{T}_0 \simeq \Bbbk^{G\times G} \#_\sigma \Bbbk G$ is the crossed product corresponding to the right regular action of G on $G \times G$ and the cocycle $\sigma(g,h) = \sum_{a,b\in G} \sigma_b^a(g,h) e_b^a$, $g,h \in G$, where $e_b^a \in \Bbbk^{G\times G}$ are the canonical idempotents and $\sigma_b^a(g,h) := \dfrac{\omega(a,g,h)}{\omega(b,g,h)}$. The comultiplication in the canonical basis of $\Bbbk^{G\times G}\#_\sigma \Bbbk G$ is determined by

$$\Delta(e_b^a \# g) = \sum_{c\in G} e_c^a \# g \otimes e_b^c \# g, \tag{4.3}$$

for all $a,b,g \in G$.

Consider the quasi-Hopf algebra structure on $H = \Bbbk^G$ with associator ω. The representation category of H is exactly Vec_ω^G. By the results of Hausser and Nill [HN], there is a tensor equivalence $\mathrm{Vec}_\omega^G \to {}_H\mathcal{M}_H^H$, where ${}_H\mathcal{M}_H^H$ is the category of quasi-Hopf bimodules with tensor product $\otimes_H$. In particular, the forgetful functor ${}_H\mathcal{M}_H^H \to {}_H\mathcal{M}_H$ is monoidal, hence a fiber funtor. The defining relations for the category ${}_H\mathcal{M}_H^H$ can be used to reconstruct a multiplication and comultiplication in the vector space $L = \Bbbk^{(G\times G)} \otimes \Bbbk G$, which make L into a weak Hopf algebra with basis H (hence a face algebra, since H is commutative) such that ${}_H\mathcal{M}_H^H \simeq \operatorname{Rep} L$. Moreover, the formulas thus obtained give the crossed product algebra structure $L = \Bbbk^{G\times G}\#_\sigma \Bbbk G$ with comultiplication (4.3). This establishes the proposition. $\square$

4.5. The categories ${}_F\mathrm{Vec}_F^G$.

We assume in this subsection that the cocycle ω is trivial. Consider the double groupoid

$$\mathcal{T} = \begin{matrix} F\times F\times G & \rightrightarrows & F \\ \downdownarrows & & \downdownarrows, \\ G & \rightrightarrows & * \end{matrix}$$

where $*$ is a set with a single element, and

- $F\times F\times G \rightrightarrows F$ is the direct product of the coarse groupoid on F and group G,
- $F\times F\times G \rightrightarrows G$ is the transformation groupoid associated to the action $. : G\times F\times F \to G$, $g.(x,y) = x^{-1}gy$.

The box in $\mathcal{T}$ corresponding to a triple (x,y,g), $x,y \in F$, $g\in G$ can be depicted as

$$x \overset{g}{\underset{h}{\square}} y,$$

where $h \in G$, $gy = xh$. Horizontal and vertical compositions are given by

$$x\,\underset{h}{\overset{g}{\square}}\,y\;y\,\underset{t}{\overset{s}{\square}}\,z = x\,\underset{ht}{\overset{gs}{\square}}\,z, \qquad \begin{array}{c} x\,\underset{h}{\overset{g}{\square}}\,y \\ x'\,\underset{t}{\overset{h}{\square}}\,y' \end{array} = xx'\,\underset{t,}{\overset{g}{\square}}\,yy',$$

for all $x, y, z, x', y' \in F$, $g, h, s, t \in G$. In this example the core groupoid $\mathbf{D}$ is isomorphic to F, and the right corner function is given by the formula

$$\neg\left(x\,\underset{h}{\overset{g}{\square}}\,y\right) = |F|. \tag{4.4}$$

$\mathcal{T}$ is the *comma double groupoid* associated to the inclusion $F \to G$ [BMa, Example 1.8]. In the terminology *loc. cit.* $\mathcal{T}$ coincides with the transformation double groupoid $\mathcal{T}(F) \ltimes (\chi_G, \mathrm{id})$, where $\mathcal{T}(F)$ is the double groupoid in Subsection 4.2, and χ_G is the anchor map corresponding to G.

Proposition 4.7. $\Bbbk\mathcal{T} \simeq {}_F\operatorname{Vec}_F^G$ *as tensor categories.*

Proof. Since the vertical groupoid $\mathcal{B} \rightrightarrows \mathcal{H}$ is a transformation groupoid, $\Bbbk\mathcal{T} \simeq \Bbbk^G \# \Bbbk(F \times F^{\mathrm{op}})$ as an algebra, where the right hand side expression is the smash product corresponding to the action $\Bbbk(F \times F^{\mathrm{op}}) \otimes \Bbbk^G \to \Bbbk^G$, $\langle (x, y).f, g\rangle := \langle f, g.(x, y)\rangle$. Then there is an equivalence $\Bbbk\mathcal{T} \simeq {}_F\operatorname{Vec}_F^G$ of $\Bbbk$-linear categories. Under this identification, the comultiplication on $\Bbbk\mathcal{T}$ is given by

$$\Delta(e_g \#(x \otimes y)) = \sum_{st=g} \left(e_s \# x \otimes e^{(1)}\right) \otimes \left(e_t \# e^{(2)} \otimes y\right),$$

where $e_g \in \Bbbk^G$ are the canonical idempotents, $g \in G$, and $e = e^{(1)} \otimes e^{(2)} \in \Bbbk F \otimes (\Bbbk F)^{\mathrm{op}}$ is the separability idempotent given by $e = \sum_{x \in F} \dfrac{1}{|F|} x \otimes x^{-1}$.

In particular, $\Delta(1) = \sum_{s,t} \left(e_s \# 1_G \otimes e^{(1)}\right) \otimes \left(e_t \# e^{(2)} \otimes 1_G\right)$. Therefore, as in the proof of Lemma 4.3, we see that the natural equivalence $\Bbbk\mathcal{T} \simeq {}_F\operatorname{Vec}_F^G$ of $\Bbbk$-linear categories preserves tensor products. This finishes the proof of the proposition. $\square$

References

[AN] N. Andruskiewitsch and S. Natale, *Double categories and quantum groupoids*, preprint math.QA/0308228 (2003), Publ. Mat. Urug., to appear.

[BNSz] G. Böhm, F. Nill and K. Szlachányi, *Weak Hopf algebras I. Integral theory and C^*-structure*, J. Algebra **221**, 385–438 (1999).

[BSz] G. Böhm and K. Szlachányi, *A coassociative C^*-quantum group with nonintegral dimensions*, Lett. in Math. Phys. **35**, 437–456 (1996).

[BSz2] G. Böhm and K. Szlachányi, *Weak Hopf algebras. II: Representation theory, dimensions, and the Markov trace*, J. Algebra **233**, 156–212 (2000).

[B] R. Brown, *Crossed complexes and homotopy groupoids as non commutative tools for higher dimensional local-to-global problems*, Fields Inst. Commun. **43**, 101–130, Amer. Math. Soc., (2004).

[BMa] R. Brown and K. Mackenzie, *Determination of a double Lie groupoid by its core diagram*, J. Pure Appl. Algebra **80**, 237–272 (1992).

[BS] R. Brown and C. Spencer, *Double groupoids and crossed modules*, Cahiers Topo. et Géo. Diff. **XVII**, 343–364 (1976).

[E] C. Ehresmann, *Catégories doubles et catégories structures*, C. R. Acad. Sci. Paris **256**, 1198–1201 (1963).

[ENO] P. Etingof, D. Nikshych and V. Ostrik, *On fusion categories*, preprint math.QA/0203060 (2002).

[HN] F. Hausser and F. Nill, *Integral theory for quasi-Hopf algebras*, preprint math.QA/9904164 (1999).

[H] T. Hayashi, *A brief introduction to face algebras*, in “New trends in Hopf Algebra Theory”; Contemp. Math. **267** (2000), 161–176.

[Ma1] K. Mackenzie, *Double Lie algebroides and Second-order Geometry, I*, Adv. Math. **94**, 180–239 (1992).

[Ma2] K. Mackenzie, *Double Lie algebroides and Second-order Geometry, II*, Adv. Math. **154**, 46–75 (2000).

[Mj] S. Majid, *Foundations of quantum group theory*, Cambridge Univ. Press, Cambridge (1995).

[Na] S. Natale, *Frobenius-Schur indicators for a class of fusion categories*, preprint math.QA/0312466 (2003), Pacific J. Math., to appear.

[N1] D. Nikshych, *On the structure of weak Hopf algebras*, Adv. Math. **170**, 257–286 (2002).

[N2] D. Nikshych, *Semisimple weak Hopf algebras*, J. Algebra **275**, 639–667 (2004).

[NV] D. Nikshych and L. Vainerman, *Finite quantum groupoids and their applications*, Math. Sci. Res. Inst. Publ. **43**, 211–262 (2002).

[O] V. Ostrik, *Module categories over the Drinfeld double of a finite group*, Int. Math. Res. Not. **2003**, 1507–1520 (2003), preprint math.QA/0202130.

[T] M. Takeuchi, *Survey on matched pairs of groups. An elementary approach to the ESS-LYZ theory*, Banach Center Publ. **61**, 305-331 (2003).

Facultad de Matemática, Astronomía y Física, Universidad Nacional de Córdoba. CIEM – CONICET. (5000) Ciudad Universitaria, Córdoba, Argentina

E-mail address: `andrus@mate.uncor.edu,`

URL: `http://www.mate.uncor.edu/andrus`

E-mail address: `natale@mate.uncor.edu,`

URL: `http://www.mate.uncor.edu/natale`